Exam Quantum Probability

May 28nd, 2025

Hints: You can use all results proved in the lecture notes (without proving them yourselves), as well as claims one is supposed to prove in exercises from the lecture notes. You can also use a claim you are supposed to prove in one excercise below to solve another excercise (even if you did not prove the claim). Partial solutions also yield points.

Exercise 1 (A non-ideal measurement) Let $\mathcal{H}$ be a complex inner product space with orthonormal basis $\{e(0), e(1)\}$ and let $P_i := |e(i)\rangle\langle e(i)|$ denote the orthogonal projection on the i-th basis vector ($i = 0, 1$). For each $\theta \in [0, \pi/2]$, we define hermitian operators $V_\theta(0)$ and $V_\theta(1)$ in $\mathcal{L}(\mathcal{H})$ by

$$V_\theta(0) := \cos(\theta)P_0 + \sin(\theta)P_1 \quad \text{and} \quad V_\theta(1) := \sin(\theta)P_0 + \cos(\theta)P_1,$$

and we define $T_\theta : \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by

$$T_\theta(A \otimes B) := \sum_{i=0}^{1} \langle e(i)|B|e(i)\rangle V_\theta(i) A V_\theta(i) \qquad \big(A, B \in \mathcal{L}(\mathcal{H})\big).$$

(a) Show that T_θ is of the form

$$T_\theta(A \otimes B) := \sum_{i=0}^{1} W_\theta(i)(A \otimes B) W_\theta(i)^* \qquad \big(A, B \in \mathcal{L}(\mathcal{H})\big),$$

where $W_\theta(i) : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is given by

$$W_\theta(i)\big(\psi(0) \otimes \psi(1)\big) := \langle e(i)|\psi(1)\rangle V_\theta(i)\psi(0) \qquad \big(\psi(0), \psi(1) \in \mathcal{H}\big).$$

(b) Show that the dual map $T'_\theta : \mathcal{L}(\mathcal{H})' \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H})'$ is an operation.

(c) Let ρ be a state on $\mathcal{L}(\mathcal{H})$. Show that the first marginal of $T'_\theta\rho$ is given by

$$S'_\theta\rho(A) := (T'_\theta\rho)(A \otimes 1) = \sum_{i=0}^{1} \rho\big(V_\theta(i) A V_\theta(i)\big) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).$$

(d) Show that $S'_{\pi/4}\rho = \rho$ for each state ρ on $\mathcal{L}(\mathcal{H})$.

(e) Let ρ be the state on $\mathcal{L}(\mathcal{H})$ defined as

$$\rho(A) := \sum_{i=0}^{1} \tfrac{1}{2}\langle e(i)|A|e(i)\rangle \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).$$

Calculate the probabilities

$$(T'_\theta\rho)(P_i \otimes P_j) \qquad (0 \leq i, j \leq 1).$$

Exercise 2 (A little bit of eavesdropping) Let $\mathcal{H}$ be a complex inner product space with orthonormal basis $\{e(0), e(1)\}$. For each $\alpha \in \mathbb{R}$, let $\{e_\alpha(0), e_\alpha(1)\}$ be the orthonormal basis given by

$$e_\alpha(0) := \cos(\alpha)e(0) + \sin(\alpha)e(1) \quad \text{and} \quad e_\alpha(1) := e_{\alpha+\pi/2}(0) = -\sin(\alpha)e(0) + \cos(\alpha)e(1).$$

We set $e_\alpha := e_\alpha(0)$ and observe that $e_\alpha(1) = e_{\alpha+\pi/2}$. We set $P_\alpha := |e_\alpha\rangle\langle e_\alpha|$ and let

$$\psi := \frac{1}{\sqrt{2}}\big(e_\alpha(0) \otimes e_\alpha(0) + e_\alpha(1) \otimes e_\alpha(1)\big)$$

be the entangled state from formula (7.3) of the lecture notes. It has been proved there that this definition does not depend on the angle α. Let σ be the pure state

$$\sigma(A) := \rho_\psi(A) = \langle\psi|A|\psi\rangle \qquad \big(A \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})\big).$$

As in Section 9.2 of the lecture note, Alice prepares two fotons in the state σ, keeps the first one for herself, and sends the second one to Bob.

(a) Let ρ, defined as

$$\rho(A) := \sigma(1 \otimes A) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big),$$

denote the second marginal of σ. Show that

$$\rho(A) := \sum_{i=0}^{1} \tfrac{1}{2}\langle e(i)|A|e(i)\rangle \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).$$

Eve is eavesdropping on Alice and Bob. She performs the operation T'_θ of Exercise 1 on Bob's foton, creating a second foton for herself that in view of part (e) of Exercise 1 is positively correlated with Bob's foton as long as $\theta \in [0, \pi/4]$. If $\theta = 0$, then Eve has full information about the secret code that Bob receives, while in the other extreme $\theta = \pi/4$ Eve has no information at all. By choosing $0 < \theta < \pi/4$, Eve hopes to get at least some information about the secret code without being detected. By eavesdropping, Eve disturbs the state σ. The new state is

$$\sigma'(A \otimes B) := \sigma\big(A \otimes S_\theta(B)\big) \qquad \big(A, B \in \mathcal{L}(\mathcal{H})\big),$$

where

$$S_\theta(B) := \sum_{k=0}^{1} V_\theta(k) B V_\theta(k) \qquad \big(B \in \mathcal{L}(\mathcal{H})\big),$$

and $V_\theta(0), V_\theta(1)$ are defined as in Exercise 1. For $\alpha, \beta \in \mathbb{R}$, we define

$$\pi_{\alpha,\beta}(+,-) := \sigma(P_\alpha \otimes P_{\beta+\pi/2}) \quad \text{and} \quad \pi'_{\alpha,\beta}(+,-) := \sigma'(P_\alpha \otimes P_{\beta+\pi/2}).$$

(b) Show that

$$\pi'_{\alpha,\beta}(+,-) = \tfrac{1}{2}\sum_{k=0}^{1} \|P_{\beta+\pi/2}V_\theta(k)e_\alpha\|^2 \qquad (\alpha,\beta \in \mathbb{R}).$$

It has been shown in the lecture notes that

$$\pi_{0,0}(+,-) = 0, \quad \pi_{\gamma,0}(+,-) = \pi_{0,-\gamma}(+,-) = \tfrac{1}{2}\sin(\gamma)^2, \quad \pi_{\gamma,-\gamma}(+,-) = \tfrac{1}{2}\sin(2\gamma)^2.$$

Alice and Bob use the fact that $\pi_{0,0}(+,-) = 0$ to send the secret code without transmission errors. Moreover, they used the fact that for $\gamma = \pi/6$,

$$\pi_{\gamma,0}(+,-) + \pi_{0,-\gamma}(+,-) - \pi_{\gamma,-\gamma}(+,-) = -\tfrac{1}{8} < 0$$

to check that the state σ is entangled, which they view as proof that Eve has not been eavesdropping.

(c) Calculate

$$\pi'_{0,0}(+,-), \quad \pi'_{\gamma,0}(+,-), \quad \pi'_{0,-\gamma}(+,-), \quad \text{and} \quad \pi'_{\gamma,-\gamma}(+,-).$$

Hint: First derive a formula for $\pi'_{\alpha,\beta}(+,-)$. You can use that $P_\alpha e_\beta = \cos(\beta-\alpha)e_\alpha$ $(\alpha,\beta \in \mathbb{R})$.

(d) Prove that σ' is entangled if $\cos(\theta - \pi/4)^2 > \sqrt{2/3}$.

Solutions

Ex 1

(a) The adjoint $W_\theta(i)^*$ of $W_\theta(i)$ is the operator $W_\theta(i)^* : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ that is uniquely characterised by the fact that for each $\phi, \psi(0), \psi(1) \in \mathcal{H}$

$$\begin{aligned}&\langle \psi(0) \otimes \psi(1) | W_\theta(i)^* \phi\rangle = \langle W_\theta(i)\big(\psi(0) \otimes \psi(1)\big)|\phi\rangle = \langle e(i)|\psi(1)\rangle^* \langle V_\theta(i)\psi(0)|\phi\rangle \\ &\quad = \langle \psi(1)|e(i)\rangle\langle\psi(0)|V_\theta(i)\phi\rangle = \langle \psi(0) \otimes \psi(1)|V_\theta(i)\phi \otimes e(i)\rangle.\end{aligned}$$

Since this holds for all $\psi(0), \psi(1) \in \mathcal{H}$, we conclude that

$$W_\theta(i)^*\phi = V_\theta(i)\phi \otimes e(i) \qquad \big(\phi \in \mathcal{H},\ i = 0, 1\big).$$

It follows that for each $\phi \in \mathcal{H}$ and $A, B \in \mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned}&\sum_{i=0}^{1} W_\theta(i)(A \otimes B)W_\theta(i)^*\phi = \sum_{i=0}^{1} W_\theta(i)(A \otimes B)(V_\theta(i)\phi \otimes e(i)) \\ &\quad = \sum_{i=0}^{1} W_\theta(i)(AV_\theta(i)\phi \otimes Be(i)) = \sum_{i=0}^{1}\langle e(i)|B|e(i)\rangle V_\theta(i)AV_\theta(i)\phi = T_\theta(A \otimes B).\end{aligned}$$

(b) By part (a) and Stinespring's theorem, T_θ is a completely positive map. It remains to show that $T_\theta(1 \otimes 1) = 1$, where $1 \otimes 1$ is the identity in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$. We calculate

$$\begin{aligned}T_\theta(1 \otimes 1) &= \sum_{i=0}^{1}\langle e(i)|1|e(i)\rangle V_\theta(i)1V_\theta(i) = \sum_{i=0}^{1} V_\theta(i)V_\theta(i) \\ &= \big(\cos(\theta)P_0 + \sin(\theta)P_1\big)^2 + \big(\sin(\theta)P_0 + \cos(\theta)P_1\big)^2 \\ &= \big(\cos(\theta)^2P_0 + \sin(\theta)^2P_1\big) + \big(\sin(\theta)^2P_0 + \cos(\theta)^2P_1\big) = P_0 + P_1 = 1.\end{aligned}$$

(c) Immediate from the definition of T_θ, since

$$T_\theta(A \otimes 1) := \sum_{i=0}^{1}\langle e(i)|1|e(i)\rangle V_\theta(i)AV_\theta(i) = \sum_{i=0}^{1} V_\theta(i)AV_\theta(i) =: S_\theta(A) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big),$$

which then gives

$$\begin{aligned}(T'_\theta\rho)(A \otimes 1) &= \rho\big(T_\theta(A \otimes 1)\big) = \rho\big(S_\theta(A)\big) \\ &= \rho\Big(\sum_{i=0}^{1} V_\theta(i)AV_\theta(i)\Big) = \sum_{i=0}^{1}\rho\big(V_\theta(i)AV_\theta(i)\big) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).\end{aligned}$$

(d) We observe that

$$\begin{aligned}V_{\pi/4}(0) &= (\cos(\pi/4)P_0 + \sin(\pi/4)P_1) = \frac{1}{\sqrt{2}}(P_0 + P_1) = \frac{1}{\sqrt{2}}1, \\ V_{\pi/4}(1) &= (\sin(\pi/4)P_0 + \cos(\pi/4)P_1) = \frac{1}{\sqrt{2}}(P_0 + P_1) = \frac{1}{\sqrt{2}}1.\end{aligned}$$

As a result

$$S'_{\pi/4}\rho(A) = \tfrac{1}{2}\sum_{i=0}^{1}\rho(1A1) = \rho(A) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).$$

(e)

$$(T'_\theta\rho)(P_i \otimes P_j) = \rho\big(T_\theta(P_i \otimes P_j)\big) = \sum_{k=0}^{1}\langle e(k)|P_j|e(k)\rangle\rho\big(V_\theta(k)P_iV_\theta(k)\big) = \rho\big(V_\theta(j)P_iV_\theta(j)\big).$$

It follows that

$$\begin{aligned}
(T'_\theta\rho)(P_0 \otimes P_0) &= \rho\big((\cos(\theta)P_0 + \sin(\theta)P_1)P_0(\cos(\theta)P_0 + \sin(\theta)P_1)\big) = \cos(\theta)^2\rho(P_0),\\
(T'_\theta\rho)(P_0 \otimes P_1) &= \rho\big((\sin(\theta)P_0 + \cos(\theta)P_1)P_0(\sin(\theta)P_0 + \cos(\theta)P_1)\big) = \sin(\theta)^2\rho(P_0),\\
(T'_\theta\rho)(P_1 \otimes P_0) &= \rho\big((\cos(\theta)P_0 + \sin(\theta)P_1)P_1(\cos(\theta)P_0 + \sin(\theta)P_1)\big) = \sin(\theta)^2\rho(P_1),\\
(T'_\theta\rho)(P_1 \otimes P_1) &= \rho\big((\sin(\theta)P_0 + \cos(\theta)P_1)P_1(\sin(\theta)P_0 + \cos(\theta)P_1)\big) = \cos(\theta)^2\rho(P_1).
\end{aligned}$$

Here

$$\rho(P_i) := \tfrac{1}{2}\sum_{j=0}^{1}\tfrac{1}{2}\langle e(j)|P_i|e(j)\rangle = \tfrac{1}{2} \quad (i = 0, 1),$$

which gives

$$\begin{aligned}
(T'_\theta\rho)(P_0 \otimes P_0) &= \tfrac{1}{2}\cos(\theta)^2,\\
(T'_\theta\rho)(P_0 \otimes P_1) &= \tfrac{1}{2}\sin(\theta)^2,\\
(T'_\theta\rho)(P_1 \otimes P_0) &= \tfrac{1}{2}\sin(\theta)^2,\\
(T'_\theta\rho)(P_1 \otimes P_1) &= \tfrac{1}{2}\cos(\theta)^2.
\end{aligned}$$

Ex 2

(a) This follows by writing

$$\begin{aligned}
\rho(A) = \sigma(1 \otimes A) = \langle\psi|1 \otimes A|\psi\rangle &= \tfrac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\langle e(i) \otimes e(i)|1 \otimes A|e(j) \otimes e(j)\rangle\\
&= \tfrac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\langle e(i)|e(j)\rangle\langle e(i)|A|e(j)\rangle = \tfrac{1}{2}\sum_{i=0}^{1}\langle e(i)|A|e(i)\rangle = \rho(A) \qquad \big(A \in \mathcal{L}(\mathcal{H})\big).
\end{aligned}$$

(b) We have

$$\begin{aligned}
\pi'_{\alpha,\beta}(+,-) &= \sigma'(P_\alpha \otimes P_{\beta+\pi/2}) = \sigma\big(P_\alpha \otimes S_\theta(P_{\beta+\pi/2})\big) \\
&= \sum_{k=0}^{1} \langle \psi | P_\alpha \otimes V_\theta(k) P_{\beta+\pi/2} V_\theta(k) | \psi \rangle \\
&= \tfrac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \langle e_\alpha(i) \otimes e_\alpha(i) | P_\alpha \otimes V_\theta(k) P_{\beta+\pi/2} V_\theta(k) | e_\alpha(j) \otimes e_\alpha(j) \rangle \\
&= \tfrac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \langle P_\alpha e_\alpha(i) \otimes P_{\beta+\pi/2} V_\theta(k) e_\alpha(i) | P_\alpha e_\alpha(j) \otimes P_{\beta+\pi/2} V_\theta(k) e_\alpha(j) \rangle \\
&= \tfrac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \langle P_\alpha e_\alpha(i) | P_\alpha e_\alpha(j) \rangle \langle P_{\beta+\pi/2} V_\theta(k) e_\alpha(i) | P_{\beta+\pi/2} V_\theta(k) e_\alpha(j) \rangle \\
&= \tfrac{1}{2} \sum_{k=0}^{1} \langle P_{\beta+\pi/2} V_\theta(k) e_\alpha(0) | P_{\beta+\pi/2} V_\theta(k) e_\alpha(0) \rangle = \tfrac{1}{2} \sum_{k=0}^{1} \| P_{\beta+\pi/2} V_\theta(k) e_\alpha \|^2.
\end{aligned}$$

(c) Using the hint, we have

$$\begin{aligned}
P_{\beta+\pi/2} V_\theta(0) e_\alpha &= P_{\beta+\pi/2}\big(\cos(\theta) P_0 + \sin(\theta) P_{\pi/2}\big) e_\alpha \\
&= P_{\beta+\pi/2}\big(\cos(\theta)\cos(\alpha) e_0 + \sin(\theta)\cos(\alpha - \pi/2) e_{\pi/2}\big) \\
&= \big(\cos(\theta)\cos(\alpha)\cos(\beta+\pi/2) + \sin(\theta)\cos(\alpha-\pi/2)\cos(\beta)\big) e_{\beta+\pi/2}
\end{aligned}$$

and

$$\begin{aligned}
P_{\beta+\pi/2} V_\theta(1) e_\alpha &= P_{\beta+\pi/2}\big(\sin(\theta) P_0 + \cos(\theta) P_{\pi/2}\big) e_\alpha \\
&= P_{\beta+\pi/2}\big(\sin(\theta)\cos(\alpha) e_0 + \cos(\theta)\cos(\alpha - \pi/2) e_{\pi/2}\big) \\
&= \big(\sin(\theta)\cos(\alpha)\cos(\beta+\pi/2) + \cos(\theta)\cos(\alpha-\pi/2)\cos(\beta)\big) e_{\beta+\pi/2}.
\end{aligned}$$

Using part (b), it follows that

$$\begin{aligned}
\pi'_{\alpha,\beta}(+,-) &= \tfrac{1}{2}\big(\cos(\theta)\cos(\alpha)\cos(\beta+\pi/2) + \sin(\theta)\cos(\alpha-\pi/2)\cos(\beta)\big)^2 \\
&\quad + \tfrac{1}{2}\big(\sin(\theta)\cos(\alpha)\cos(\beta+\pi/2) + \cos(\theta)\cos(\alpha-\pi/2)\cos(\beta)\big)^2 \\
&= \tfrac{1}{2}\big(-\cos(\theta)\cos(\alpha)\sin(\beta) + \sin(\theta)\sin(\alpha)\cos(\beta)\big)^2 \\
&\quad + \tfrac{1}{2}\big(-\sin(\theta)\cos(\alpha)\sin(\beta) + \cos(\theta)\sin(\alpha)\cos(\beta)\big)^2.
\end{aligned}$$

It follows immediately that $\pi'_{0,0}(+,-) = 0$. This means that Bob still receives Alice's signal undisturbed by Eve's eavesdropping. Next, we see that

$$\begin{aligned}
\pi'_{\gamma,0}(+,-) &= \tfrac{1}{2}\big(\sin(\theta)\sin(\gamma)\big)^2 + \tfrac{1}{2}\big(\cos(\theta)\sin(\gamma)\big)^2 = \tfrac{1}{2}\sin(\gamma)^2, \\
\pi'_{0,-\gamma}(+,-) &= \tfrac{1}{2}\big(\cos(\theta)\sin(\gamma)\big)^2 + \tfrac{1}{2}\big(\sin(\theta)\sin(\gamma)\big)^2 = \tfrac{1}{2}\sin(\gamma)^2,
\end{aligned}$$

just like in the case where Eve has not been eavesdropping. Finally, we have

$$\begin{aligned}
\pi'_{\gamma,-\gamma}(+,-) &= \tfrac{1}{2}\big(\cos(\theta)+\sin(\theta)\big)^2\big(\cos(\gamma)\sin(\gamma)\big)^2 \\
&\quad + \tfrac{1}{2}\big(\sin(\theta)+\cos(\theta)\big)^2\big(\sin(\gamma)\cos(\gamma)\big)^2 \\
&= \big(\cos(\theta)+\sin(\theta)\big)^2\big(\cos(\gamma)\sin(\gamma)\big)^2 \\
&= \big(\sqrt{2}\cos(\theta-\pi/4)\big)^2 \cdot \big(\tfrac{1}{2}\sin(2\gamma)\big)^2 \\
&= \tfrac{1}{2}\cos(\theta-\pi/4)^2\sin(2\gamma)^2.
\end{aligned}$$

In Exercise 1 (d) and (e) we have already seen that the case $\gamma = \pi/4$ corresponds to no eavesdropping so it makes sense that in this case we find the same as before. In all other cases $\pi'_{\gamma,-\gamma}(+,-)$ is reduced giving Alice and Bob a chance to notice Eve's activities.

(d) Setting $\gamma = \pi/6$ yields $\sin(\gamma) = 1/2$ and $\sin(2\gamma) = \frac{1}{2}\sqrt{3}$. Using moreover that $\cos(\theta - \pi/4) < \cos(\pi/4) = 1/\sqrt{2}$ for all $0 < \theta \leq \pi/4$, we see that

$$\begin{aligned}
&\pi'_{\gamma,0}(+,-) + \pi'_{0,-\gamma}(+,-) - \pi'_{\gamma,-\gamma}(+,-) = \sin(\gamma)^2 - \tfrac{1}{2}\cos(\theta-\pi/4)^2\sin(2\gamma)^2 \\
&\quad = \tfrac{1}{4} - \tfrac{1}{2}\cos(\theta-\pi/4)^2\tfrac{3}{4} = \tfrac{1}{8}\big(2 - 3\cos(\theta-\pi/4)^2\big),
\end{aligned}$$

which is strictly negative as long as $\cos(\theta - \pi/4)^2 > \sqrt{2/3}$. We can then apply Wigner's inequality (Lemma 9.2.1 in the lecture notes) to conclude that σ' is entangled.