# Addentum to: The Brownian net


Rongfeng Sun
EURANDOM
P.O. Box 513
5600MB Eindhoven
The Netherlands
e-mail: sun@math.tu-berlin.de

Jan M. Swart
ÚTIA
Pod vodárenskou věží 4
18208 Praha 8
Czech Republic
e-mail: swart@utia.cas.cz


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## 1 On the branching-coalescing point set

Lemmas 1 and 2 below generalize Lemmas 42, respectively, 43 in Section 8 of [SS06] and solve a problem posed there (see remark after the proof of Lemma 43). The conjecture as formulated there is false, but a slightly more careful reformulation is correct. Below, all references to theorems and such are from [SS06].

**Lemma 1 (Hopping paths starting from a closed set)**
*Let $\mathcal{N}$ be the Brownian net. Let $K \subset \mathbb{R}^2$ be closed. Let $\mathcal{D}^{\rm l}, \mathcal{D}^{\rm r} \subset \mathbb{R}^2$ be deterministic countable dense sets such that, moreover, $\mathcal{D}^{\rm l} \cap K$ is dense in $K$. Then*

$$\mathcal{N}(K) \subset \overline{\Pi(K) \cap \mathcal{H}_{\rm cros}(\mathcal{W}^{\rm l}(\mathcal{D}^{\rm l}) \cup \mathcal{W}^{\rm r}(\mathcal{D}^{\rm r}))}. \tag{1.1}$$

**Proof** By the dual characterization of the Brownian net (Theorem 10), it suffices to show that any path $\pi$ starting at some point $z = (x, t) \in K$ that does not enter wedges from outside can be approximated by paths in $\mathcal{H}_{\rm cros}(\mathcal{W}^{\rm l}(\mathcal{D}^{\rm l}) \cup \mathcal{W}^{\rm r}(\mathcal{D}^{\rm r}))$, also starting in $K$. By the compactness of $\mathcal{N}$, it suffices to show that for each $t < t_1 < \cdots < t_n$, and $0 < \varepsilon < t_1 - t$, there exists a path $\pi^\varepsilon$ started at some time in $(t - \varepsilon, t + \varepsilon)$, such that $|\pi^\varepsilon(t_i) - \pi(t_i)| \leq \varepsilon$ for all $i = 1, \ldots, n$. We use the steering argument from the proof of Lemma 30. We construct a 'fish-trap' with left and right boundary $\hat{r}, \hat{l}$ as in Figure 7. Set $S := \sup\{s < t_n : \hat{r}(s) = \hat{l}(s)\}$. For any $a, b \in \mathbb{R}$ with $a < b \leq \hat{\sigma}_{\hat{r}} \wedge \hat{\sigma}_{\hat{l}}$, define an open set $V_{(a,b)}$ by

$$V_{(a,b)} := \big\{(x, s) \in \mathbb{R}^2 : a < s < b,\ S < s,\ \hat{r}(s) < x < \hat{l}(s)\big\}. \tag{1.2}$$

We need to show that there exists a path $\pi^\varepsilon$, started at some time in $(t - \varepsilon, t + \varepsilon)$, that stays between $\hat{r}$ and $\hat{l}$. This will follow from the same arguments as in the proof of Lemma 30, provided that the set $V_{(t-\varepsilon,t+\varepsilon)} \cap \mathcal{D}^{\rm l}$ is nonempty. Since $\mathcal{D}^{\rm l} \cap K$ is dense in $K$ and $V_{(t-\varepsilon,t+\varepsilon)}$ is open, it suffices to show that $V_{(t-\varepsilon,t+\varepsilon)} \cap K$ is nonempty. Assume that this is not the case. Then, by Lemma 21 (b), we can find $a < b$ and $\hat{r} \in \hat{\mathcal{W}}^{\rm r}(\mathcal{D}^{\rm r})$, $\hat{l} \in \hat{\mathcal{W}}^{\rm r}(\mathcal{D}^{\rm l})$ starting at times $\hat{\sigma}_{\hat{r}}, \hat{\sigma}_{\hat{l}} > b$, such that $V_{(a,b)} \cap K = \emptyset$ but $z \in \partial V_{(a,b)}$, where we define

$$\partial V_{(a,b)} := \big\{(x, s) \in \mathbb{R}^2 : a < s < b,\ S \leq s,\ x = \hat{r}(s) \text{ or } x = \hat{l}(s)\big\}. \tag{1.3}$$

We claim that this is impossible. More precisely, we claim that if $K \subset \mathbb{R}^2$ is a deterministic closed set and $\hat{r} \in \hat{\mathcal{W}}^{\rm r}$, $\hat{l} \in \hat{\mathcal{W}}^{\rm r}$ are paths with deterministic starting points, then almost surely, there exist no $a, b \in \mathbb{R}$, with $a < b < \hat{\sigma}_{\hat{r}}, \hat{\sigma}_{\hat{l}}$, such that $V_{(a,b)} \cap K = \emptyset$ and $\partial V_{(a,b)} \cap K \neq \emptyset$. It suffices to prove the statement for deterministic $a, b$. Set

$$
\begin{aligned}
Y &:= \hat{r}(b) + \hat{l}(b), \\
Z &:= \hat{r}(b) - \hat{l}(b), \\
R &:= \big(\hat{r}(s) - \hat{r}(b)\big)_{s \in [a,b]}, \\
L &:= \big(\hat{l}(s) - \hat{l}(b)\big)_{s \in [a,b]}.
\end{aligned} \tag{1.4}
$$

We claim that for any $y \in \mathbb{R}$ and continuous functions $\omega_{\rm r}, \omega_{\rm l} : [a,b] \to \mathbb{R}$, the conditional probability

$$
\mathbb{P}\big[V_{(a,b)} \cap K = \emptyset \text{ and } \partial V_{(a,b)} \cap K \neq \emptyset \,\big|\, Y = y,\ R = \omega_{\rm r},\ L = \omega_{\rm l}\,\big] \tag{1.5}
$$

is zero. Indeed, for given $Y, R$, and $L$, there can be at most one value of $Z$ for which the event $V_{(a,b)} \cap K = \emptyset$ and $\partial V_{(a,b)} \cap K \neq \emptyset$ occurs. Since conditioned on $S < b$, which is necessary for $V_{(a,b)} \neq \emptyset$, the distribution of the random variable $Z$ is absolute continuous with respect to Lebesgue measure, the conditional probability in (1.5) is zero. Integrating over the distributions of $Y, R$, and $L$, we arrive at our result. ∎

**Lemma 2 (Almost sure continuity)**
*Let $\mathcal{N}$ be the Brownian net, and let $K_n, K \in \mathcal{K}(R^2_{\rm c})$ be deterministic sets satisfying $K_n \to K$. Assume that $(*, -\infty) \notin K$. Then $\mathcal{N}(K_n) \to \mathcal{N}(K)$ a.s.*

**Proof** Using the compactness of $\mathcal{N}$, by going to a subsequence if necessary, we may assume that $\mathcal{N}(K_n) \to \mathcal{A}$ for some compact subset $\mathcal{A} \subset \mathcal{N}$. Obviously, all paths in $\mathcal{A}$ have starting points in $K$, so $\mathcal{A} \subset \mathcal{N}(K)$. Write $K' := K \cap \mathbb{R}^2$ and $K'' := K \backslash K'$. Since $\mathcal{N}(z)$ is trivial for $z \in K''$, it is easy to see that $\mathcal{A} \supset \mathcal{N}(K'')$. We are left with the task to show $\mathcal{A} \supset \mathcal{N}(K')$. Choose a deterministic countable dense set $\mathcal{D} \subset \mathbb{R}^2$ such that, moreover, $\mathcal{D} \cap (K')$ is dense in $K'$. For each $z \in \mathcal{D} \cap K'$, choose $z_n \in K_n$ such that $z_n \to z$. Then $l_{z_n} \to l_z$. If $l_z$ crosses a path $r \in \mathcal{W}^{\rm r}$, then for $n$ large enough, $l_{z_n}$ also crosses $r$. Therefore, it is not hard to see that

$$
\mathcal{A} \supset \Pi(K') \cap \mathcal{H}_{\rm cros}(\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})). \tag{1.6}
$$

By Lemma 1, it follows that $\mathcal{A} \supset \mathcal{N}(K')$. ∎

**Remark** If $(*, -\infty) \in K$, then by Proposition 15 (iii), the conclusion of Lemma 2 still holds provided there exist $(x_n, t_n) \in K_n$ such that $t_n \to -\infty$ and $\limsup_{n \to \infty} |x_n|/|t_n| < 1$. Here some of the $(x_n, t_n)$ may be $(*, -\infty)$, with the convention that $|*|/|\infty| := 0$. As the proof of Proposition 15 (iii) shows, this condition cannot be relaxed very much.

## References


[SS06] R. Sun and J.M. Swart. The Brownian net. ArXiv: math.PR/0610625