

A note on extendable sets of colorings and rooted minors

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Abstract

DeVos and Seymour (2003) proved that for every set C of 3-colorings of a set X of vertices, there exists a plane graph G with vertices of X incident with the outer face such that a 3-coloring of X extends to a 3-coloring of G if and only if it belongs to C . We prove a generalization of this claim for k -colorings of X -rooted- K_{k+1} -minor-free K_{k+2} -minor-free graphs.

1 Introduction

For a positive integer k , a *set of k -colorings* of a set X of vertices is a set C of functions $f : X \rightarrow [k]$ closed under permutations of colors, i.e., for every permutation $\pi : [k] \rightarrow [k]$, if $f \in C$, then $\pi \circ f \in C$. For a graph G , we say that G *realizes* C if $X \subseteq V(G)$ and each function $f : X \rightarrow [k]$ belongs to C if and only if f extends to a proper k -coloring of G . Suppose that the set X is cyclically ordered. In this case, we say that C is *planarly realizable* if there exists a plane graph G realizing C such that all vertices of X are incident with the outer face of G and the order of appearance of the vertices of X in the boundary of the outer face of X matches the prescribed cyclic ordering of X .

The proofs of many coloring results in planar graphs, including the proof of the Four Color Theorem, depend on the following fact: For every $k \geq 4$, there exist many sets of k -colorings that are not planarly realizable (indeed, the planarly realizable sets must satisfy Kempe chain constraints). Briefly, these proofs are based on the existence of reducible configurations. Let K be a plane graph with a specified set X of vertices incident with the outer face of K , let X be cyclically ordered in the order the vertices appear along the boundary

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of the outer face of K , and let C_K be the set of all k -colorings of X that extend to a k -coloring of K . In the simplest form, we say that (K, X) is a *reducible configuration* if every non-empty planarly realizable set of k -colorings of X intersects C_K , or equivalently, no non-empty subset of the complement of C_K is planarly realizable.

The motivation for this definition is as follows. Suppose that G is a plane graph and Δ is a closed disk in the plane whose boundary intersects the drawing of G in a set Y of vertices. Let G_1 be the subgraph of G drawn in Δ and G_2 the subgraph of G drawn in the closure of the complement of Δ . If (G_1, Y) is isomorphic to (K, X) , we say that the configuration (K, X) *appears* in G . Since (K, X) is a reducible configuration, it follows that G is k -colorable if and only if G_2 is k -colorable. This reduces the problem of proving that G is k -colorable to the smaller graph G_2 .

On the other hand, DeVos and Seymour [DS03] proved that the situation is different for $k = 3$.

Theorem 1 (DeVos and Seymour [DS03]). *For every cyclically ordered set X of vertices, every set of 3-colorings of X is planarly realizable.*

Thus, for 3-coloring, (K, X) is a reducible configuration only if all 3-colorings of X extend to K , making the reducible configurations for 3-coloring much less prevalent. Of course, there are other variants of reducibility (e.g., instead of fully removing K , we can replace it by a smaller subgraph), but they are similarly affected.

The influential Hadwiger's conjecture states that for every positive integer k , every K_{k+1} -minor-free graph is k -colorable. The conjecture has only been confirmed for $k \leq 5$, see [RST93], but it has sparked a lot of interest in the theory of graph minors and in particular in their coloring properties. Thus, it is important to consider the limits of the method of reducible configurations in this context as well.

A *model* of a graph H in a graph G is a function μ that assigns pairwise vertex-disjoint connected subgraphs of G to vertices of H so that for every edge $uv \in E(H)$, there exists an edge of G with one end in $\mu(u)$ and the other end in $\mu(v)$. If H has a model in G , then H is a *minor* of G ; otherwise, we say that G is *H -minor-free*. For a set $X \subseteq V(G)$, we say that the model is *X -rooted* if $V(\mu(v)) \cap X \neq \emptyset$ for all $v \in V(H)$. If H has an X -rooted model in G , we say that H is an *X -rooted minor* of G ; otherwise, we say that G is *X -rooted- H -minor-free*.

Since the graph K_4 is not outerplanar and K_5 is not planar, Theorem 1 has the following consequence.

Corollary 2. *For every set X of vertices and every set C of 3-colorings of X , there exists an X -rooted- K_4 -minor-free K_5 -minor-free graph G realizing C .*

In this note, we show that this statement generalizes as follows.

Theorem 3. *For every integer $k \geq 3$, every set X of vertices and every set C of k -colorings of X , there exists an X -rooted- K_{k+1} -minor-free K_{k+2} -minor-free graph G realizing C .*

Thus, similarly to Theorem 1, reducible configurations in general cannot exploit the global properties when the number of colors is smaller than what is postulated by Hadwiger’s conjecture.

Holroyd [Hol97] proposed the following strengthening of Hadwiger’s conjecture: If G is k -colorable and every k -coloring of G uses all k colors on a set X of vertices, then K_k is an X -rooted minor of G . Holroyd proved this to be true for $k = 3$, and Martinsson and Steiner [MS24] for $k = 4$. If true in general, this conjecture implies that Theorem 3 does not extend to X -rooted- K_k -minor-free graphs, i.e., the following conjecture is a weakening of Holroyd’s one.

Conjecture 4. *For every integer $k \geq 3$, there exists a set X of vertices and a set C of k -colorings of X such that no X -rooted- K_k -minor-free graph G realizes C .*

2 Proof

Our proof depends on the result of DeVos and Seymour (Theorem 1). Since the case $k = 3$ is Corollary 2, we assume $k \geq 4$ from now on.

We are going to need several gadgets. The first one is a “copy gadget” $F_{\text{copy},k}(u, v)$, obtained from the clique on $k + 1$ vertices by deleting an edge uv . Clearly, $f : \{u, v\} \rightarrow [k]$ extends to a k -coloring of $F_{\text{copy},k}(u, v)$ if and only if $f(u) = f(v)$. Let $F_{\text{copy},k}^+ = F_{\text{copy},k}(u, v) + uv$ be the clique of size $k + 1$.

The second gadget is more complicated. For $s \in \{4, \dots, k\}$, we are going to construct $F_{\text{enc},s,k}(u, v, w; y_4, \dots, y_k)$ as a graph F that contains a set $A = \{u, v, w, y_4, \dots, y_k\}$ of distinct vertices and the following claim holds: Let $f : A \rightarrow [k]$ be a function such that $f(y_i) = i$ for all $i \in \{4, \dots, k\}$. Then f extends to a k -coloring of F if and only if

- $f(u) \in \{1, 2, 3\}$ and $f(u) = f(v) = f(w)$, or
- $f(u) = s$, $f(v), f(w) \in \{1, 2, 3\}$, and $f(v) \neq f(w)$, or
- $f(u) \in \{4, \dots, k\} \setminus \{s\}$ and $f(v) = f(w) \in \{1, 2, 3\}$.

Thus, this gadget encodes information about the color of u in the 3-coloring of v and w : The vertices v and w have different colors if and only if u has color s , and if the color of u belongs to $\{1, 2, 3\}$, it is copied to the colors of v and w .

We construct F from several pieces, see Figure 1:

- Let F'_1 be graph with vertex set $\{u, v, w, v_1, v_2, w_1, w_2\}$ and edges uv_i, v_iv, uw_i, w_iw for $i \in \{1, 2\}$, v_1v_2 , and w_1w_2 . Let F_1 be obtained from F'_1 by adding the vertices y_4, \dots, y_k adjacent to all vertices of $V(F'_1) \setminus \{u\}$. Note that f extends to a k -coloring of F_1 if and only if either
 - $f(u) \in \{4, \dots, k\}$ and $f(v), f(w) \in \{1, 2, 3\}$, or
 - $f(u) = f(v) = f(w) \in \{1, 2, 3\}$.
- Let F'_2 be the graph with vertex set $\{u, v, w, u', w_1, w_2\}$ and edges $u'w_i, w_iw$ for $i \in \{1, 2\}$, uu' , w_1w_2 , and $u'v$. Let F'_2 be obtained from F'_2 by

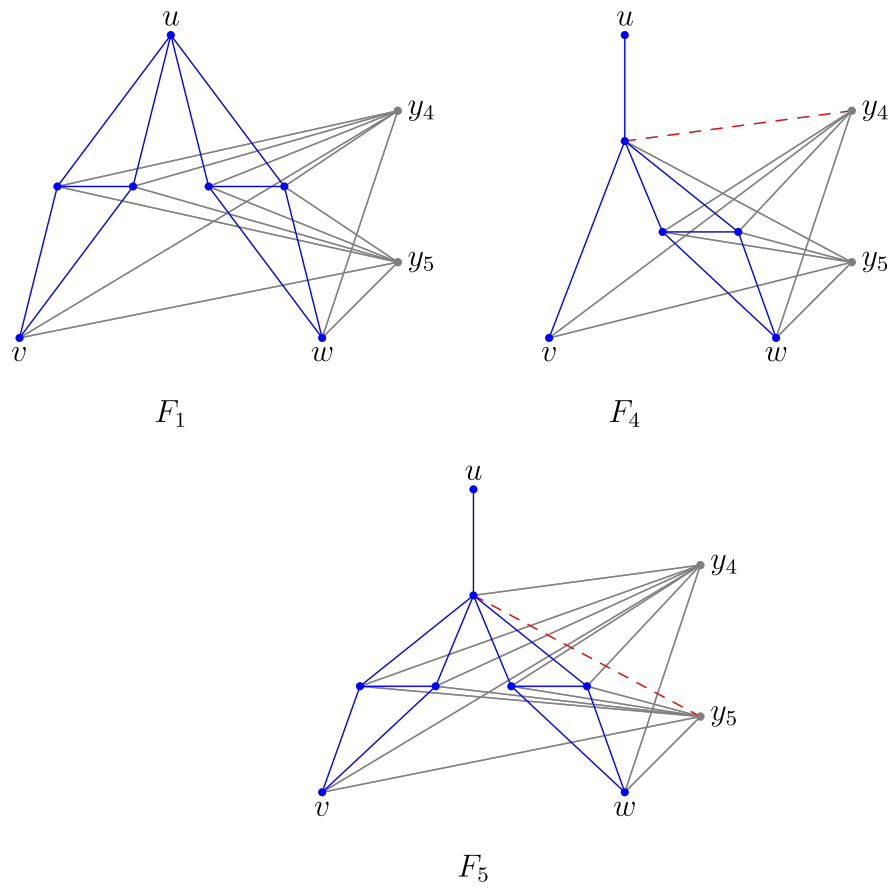


Figure 1: Pieces of the gadget $F_{\text{enc},4,5}(u, v, w; y_4, y_5)$.

adding the vertices y_4, \dots, y_k adjacent to all vertices of $V(F'_2) \setminus \{u\}$. Let $F_s = F'_2 - u'y_s$. Note that f extends to a k -coloring of F_s if and only if either

- $f(u) \neq s$ and $f(v), f(w) \in \{1, 2, 3\}$, or
- $f(u) = s$, $f(v), f(w) \in \{1, 2, 3\}$, and $f(v) \neq f(w)$.

- Let F'_3 be the graph with vertex set $\{u, v, w, u', v_1, v_2, w_1, w_2\}$ and edges $u'v_i, v_iv, u'w_i, w_iw$ for $i \in \{1, 2\}$, uu' , v_1v_2 , and w_1w_2 . Let F''_3 be obtained from F'_3 by adding the vertices y_4, \dots, y_k adjacent to all vertices of $V(F'_3) \setminus \{u\}$. For $r \in \{4, \dots, k\} \setminus \{s\}$, let $F_r = F''_3 - u'y_r$. Note that f extends to a k -coloring of F_r if and only if either

- $f(u) \neq r$ and $f(v), f(w) \in \{1, 2, 3\}$, or
- $f(u) = r$ and $f(v) = f(w) \in \{1, 2, 3\}$.

Then F is obtained from the disjoint union of copies of F_1, F_4, \dots, F_k by identifying the copies of the vertices A . Let $F_{\text{enc},s,k}^+$ be obtained from F by adding all edges of a clique on $\{u, v, w; y_4, \dots, y_k\}$.

Observation 5. *For all integers $k \geq 4$ and $s \in \{4, \dots, k\}$, the graphs $F_{\text{copy},k}^+$ and $F_{\text{enc},s,k}^+$ are K_{k+2} -minor-free.*

Proof. This is clear for $F_{\text{copy},k}^+ = K_{k+1}$. The graph of $F_{\text{enc},s,k}^+$ can be obtained as follows: We start with the planar graphs $F'_1 + uvw$, $F'_2 + uvw$, and $k-4$ copies of $F'_3 + uvw$, which are K_5 -minor-free. We take the clique-sum of these graphs on the triangle uvw , obtaining another K_5 -minor-free graph. Finally, we add $k-3$ vertices y_4, \dots, y_k and the incident edges, resulting in a K_{k+2} -minor-free graph. \square

We are now ready to prove the main result.

Proof of Theorem 3. Let $X = \{x_1, \dots, x_m\}$, and let $Y = \{y_4, \dots, y_k\}$ and $Z = \{z_{i,j} : i \in [m], j \in \{3, \dots, k\}\}$ be sets of distinct vertices. Given a function $f : X \rightarrow [k]$, let C_f be the set of functions $g : Z \rightarrow [3]$ such that for each $i \in [m]$,

- if $f(x_i) \in \{1, 2, 3\}$, then $g(z_{i,3}) = g(z_{i,4}) = \dots = g(z_{i,k}) = f(x_i)$, and
- if $f(x_i) = s \in \{4, \dots, k\}$, then $g(z_{i,3}) = \dots = g(z_{i,s-1}) \neq g(z_{i,s}) = \dots = g(z_{i,k})$.

A function $h : X \cup Y \cup Z \rightarrow [k]$ is Y -rainbow if $h(y_i) = i$ for $i \in \{4, \dots, k\}$. We are going to construct a graph G_1 with $X \cup Y \cup Z \subseteq V(G_1)$ such that every Y -rainbow function h can be extended to a k -coloring of G_1 if and only if its restrictions f and g to X and Z , respectively, satisfy $g \in C_f$. Let us remark that this implies that the restriction of h to Z is a 3-coloring. Thus, for each $i \in [m]$, the graph G_1 uniquely encodes the color of x_i in the 3-coloring of $\{z_{i,3}, \dots, z_{i,k}\}$.

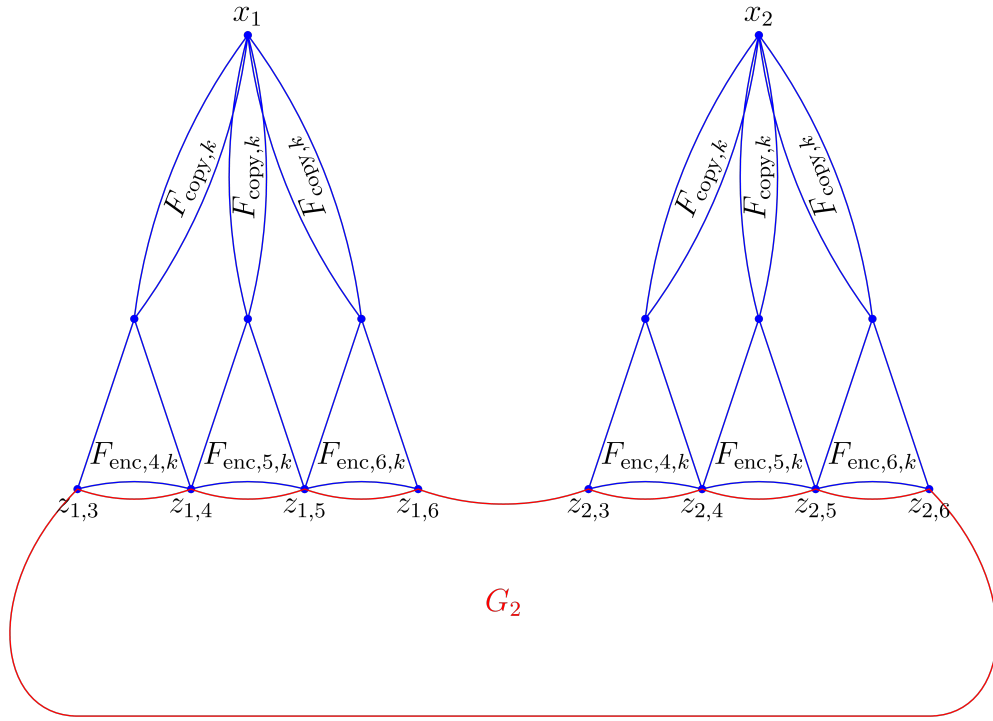


Figure 2: The construction from the proof of Theorem 3, the case $m = 2$ and $k = 6$. The apex vertices y_4, \dots, y_k are not shown.

We construct G_1 as follows, see the top part of Figure 2. We start with the set of vertices $X \cup Y \cup Z$. For $i \in [m]$ and $j \in \{4, \dots, k\}$, we add a vertex $x_{i,j}$ and a copy of the gadget $F_{\text{copy},k}(x_i, x_{i,j})$. For $i \in [m]$ and $s \in \{4, \dots, k\}$, we add a copy of the gadget $F_{\text{enc},s,k}(x_{i,s}, z_{i,s-1}, z_{i,s}; y_4, \dots, y_k)$. Then G_1 has the property just described.

Let C be the set of k -colorings of X that we want to realize and let $C' = \bigcup_{f \in C} C_f$. Observe that C' is a set of 3-colorings of Z . Let G'_2 be a plane graph with the vertices

$$z_{1,3}, z_{1,4}, \dots, z_{1,k}, z_{2,3}, \dots, z_{2,k}, \dots, z_{m,3}, \dots, z_{m,k}$$

drawn in the boundary of the outer face of G'_2 in order such that G'_2 realizes C' ; such a plane graph exists by Theorem 1. Let G_2 be the graph obtained from G'_2 by adding Y and all edges between Y and G'_2 as well as the edges of the clique on Y . We define G to be the union of G_1 and G_2 . We claim that G realizes C .

Indeed, for any k -coloring $f \in C$, we can choose a Y -rainbow function $h : X \cup Y \cup Z \rightarrow [k]$ that extends f and whose restriction g to Z satisfies $g \in C_f$. Such a function can be extended to a k -coloring of both G_1 and G_2 , and hence also of G .

Conversely, suppose that $h : X \cup Y \cup Z \rightarrow [k]$ extends to a k -coloring of G , and let f be the restriction of h to X . We need to show that $f \in C$. Since C is a set of k -colorings, we can permute the colors and without loss of generality assume that h is Y -rainbow. By the choice of G_2 , the restriction g of h to Z is a 3-coloring and $g \in C'$, and by the choice of G_1 , we have $g \in C_f$. It follows that $C_f \cap C' \neq \emptyset$, and thus $f \in C$ by the choice of C' .

It remains to argue that G is X -rooted- K_{k+1} -minor-free and K_{k+2} -minor-free. Observe that G can be obtained as follows. We start with the plane graph G_0 consisting of G'_2 and the graph drawn in Figure 2 in blue (i.e., the triangles $x_{i,j}z_{i,j-1}z_{i,j}$ and the edges $x_ix_{i,j}$ for $i \in [m]$ and $j \in \{4, \dots, k\}$). Note that the vertices of X are drawn in the boundary of the outer face of G_0 , and thus G_0 is X -rooted- K_4 -minor-free and K_5 -minor-free. We then add $k-3$ universal vertices of Y , obtaining a graph which is X -rooted- K_{k+1} -minor-free and K_{k+2} -minor free. Finally, we perform clique-sums of this graph with copies of graphs $F_{\text{copy},k}^+$ and $F_{\text{enc},\star,k}^+$. Since the added graphs are K_{k+2} -minor-free and all vertices of X are contained in G_0 , it follows that the resulting graph G is also X -rooted- K_{k+1} -minor-free and K_{k+2} -minor-free. \square

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