Monotone duality of interacting particle systems

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Abstract

In his paper from 1986 Gray developed a theory of dual processes for attractive spin systems. Based on his work Sturm and Swart systematically investigated monotonicity-based pathwise dualities for Markov processes in general and interacting particle systems in particular. In this paper we only consider monotone interacting particle systems whose state space is the Cartesian product of countably many copies of a finite set. For this class of processes Sturm and Swart only showed the well-definedness of their dual processes if started from finite initial states. This paper closes this gap and shows how to construct a well-defined pathwise dual process of a monotone interacting particle system that can also be started from an infinite initial state. This then allows us to study invariant laws of the dual process and connect them to properties of the original interacting particle system.

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1 Introduction

1.1 Aim of the paper

Spin systems are Markov processes taking values in the space $\{0,1\}^\Lambda$ of configurations $x = (x(i))_{i\in\Lambda}$ of zeros and ones on a countable set Λ (often $\Lambda = \mathbb{Z}^d$) that we will call the *grid*.[1] The value $x(i) \in \{0,1\}$ is called the *spin* at the *site* $i \in \Lambda$. In a spin system, the spins can change their value at only one site at a time. Interacting particle systems are a generalization of spin systems in which the *local state space* $\{0,1\}$ is replaced by a general finite set S. Now $x(i)$ is called the *local state* and the rule that the local state can change at only one site at a time is relaxed so that any finite number of sites can change their state at one time.

For several concrete spin systems such as the contact process and the voter model the existence of a dual process has been known since the early 1970ies. In the late 1970ies Harris [Har76, Har78] and Griffeath [Gri79] showed that each additive interacting particle system with local state space $S = \{0,1\}$ has a dual, explaining the existence of the aforementioned dual processes.[2] Later, Gray [Gra86] developed a duality theory for monotone spin systems, a class of spin systems containing the additive ones.[3] Sturm and Swart [SS18] treat both additive and monotone duality for interacting particle systems with general local state spaces S.

For additive interacting particle systems, duality is a symmetric relation in the sense that the dual process is also an additive interacting particle system. To distinguish the two, the original process is called the *forward* process, since one can think of the dual as running backwards in time. Duality allows one to make a connection between the behavior of processes started in finite and infinite initial states. Indeed, the forward process has a nontrivial upper invariant law if and only if the dual process started from a finite initial state survives with positive probability, and likewise with the roles of the forward and dual processes reversed [Har76, Theorem 10.1], [Gri79, Theorem 3.1]. For monotone interacting particle systems that are not additive, duality is no longer a symmetric relation in the sense that the state space of the dual process is not of the form S^Λ. In all work so far, the dual processes have been constructed for finite initial states only. This allows one to prove that the forward process has a nontrivial upper invariant law if and only if the dual process started from a finite initial state survives with positive probability, but not the analogue statement with the roles of the forward and dual processes reversed.

In the present paper, we fill this gap by showing that the duals of a large class of monotone interacting particle systems can be started in infinite initial states and have an upper invariant law that is nontrivial if and only if the forward process started from a finite initial state survives with positive probability.

1.2 Interacting particle systems

Let E be a compact metrizable space and let $\mathcal{M}_1(E)$ denote the space of probability measures on E, equipped with the topology of weak convergence. We recall that a *Feller semigroup* on E is a collection of probability kernels $(P_t)_{t\geq 0}$ on E such that

(i) $(x,t) \mapsto P_t(x,\,\cdot\,)$ is a continuous map from $E \times [0,\infty)$ to $\mathcal{M}_1(E)$,

(ii) $P_0 = 1$ and $P_sP_t = P_{s+t}$ $\quad (s,t \geq 0)$,

[1] This is often called the *lattice* but we reserve the latter term for its order-theoretic meaning.

[2] Harris called the dual process "associate" and adopted a stronger definition of "additive" than Griffeath, who used the term "lineal additive" for the concept considered by Harris. We will stick to Harris' stronger definition of additivity.

[3] Monotone spin systems are often called attractive spin systems since for spin systems (but not for the more general interacting particle systems), the two concepts coincide [Lig85, Theorem III.2.2].

where P_sP_t denotes the concatenation of P_s and P_t and 1 denotes the identity kernel defined as $1(x,\,\cdot\,) := \delta_x$, the delta measure on x, for each $x \in E$. A *Feller process* is a Markov process that has a Feller semigroup as its transition kernels.

Let S be a finite set, called the local state space and let Λ be a countable set, called the grid. Let S^Λ be the set of functions $x : \Lambda \to S$, equipped with the product topology. Then S^Λ is a compact metrizable space.

For each $\Delta \subset \Lambda$ and $x \in S^\Lambda$, we let $x_\Delta = (x(i))_{i\in\Delta}$ denote the restriction of x to Δ. By definition, a *local map* is a map $m : S^\Lambda \to S^\Lambda$ for which there exists a finite set $\Delta \subset \Lambda$ and a map $m' : S^\Delta \to S^\Delta$ such that

$$m(x)(i) = \begin{cases} m'(x_\Delta)(i) & \text{if } i \in \Delta, \\ x(i) & \text{else,} \end{cases} \qquad (x \in S^\Lambda,\ i \in \Lambda). \tag{1.1}$$

For each local map m, we let

$$\begin{aligned} \mathcal{D}(m) &:= \big\{ i \in \Lambda : \exists x \in S^\Lambda \text{ s.t. } m(x)(i) \neq x(i) \big\}, \\ \mathcal{R}_2(m) &:= \big\{ (j,i) \in \Lambda^2 : \exists x, y \in S^\Lambda \text{ s.t. } m(x)(i) \neq m(y)(i) \text{ and } x_{\Lambda\setminus\{j\}} = y_{\Lambda\setminus\{j\}} \big\}, \end{aligned} \tag{1.2}$$

denote the set of sites i where m can change the local state, respectively, the set of pairs of sites (j,i) such that the value of $x(j)$ is relevant for the value of $m(x)(i)$. Let $\mathcal{G}$ be a countable set of local maps and let $(r_m)_{m\in\mathcal{G}}$ be non-negative real constants that satisfy the summability condition

$$\sup_{i\in\Lambda} \sum_{m\in\mathcal{G}} r_m 1_{\mathcal{D}(m)}(i) \Big(1 + \sum_{j\in\Lambda} 1_{\mathcal{R}_2(m)}(j,i) \Big) < \infty, \tag{1.3}$$

where 1_A denotes the indicator function of a set A. For each continuous real function f on S^Λ that depends on finitely many coordinates, we define

$$Gf(x) := \sum_{m\in\mathcal{G}} r_m \big\{ f(m(x)) - f(x) \big\} \qquad (x \in S^\Lambda). \tag{1.4}$$

Then it is known [Swa22, Theorem 4.30] that G is closable and its closure generates a Feller semigroup $(P_t)_{t\geq 0}$. General theory [Kal97, Theorem 17.15] then says that for each initial law on S^Λ, there exists a unique (in law) Feller process $(X_t)_{t\geq 0}$ with values in S^Λ and càdlàg (i.e., right-continuous with left limits) sample paths such that the transition probabilities of $(X_t)_{t\geq 0}$ are given by $(P_t)_{t\geq 0}$.

It is known that $(X_t)_{t\geq 0}$ can be constructed from a graphical representation ω as follows. Let ρ be the measure on $\mathcal{G} \times \mathbb{R}$ defined by $\rho(\{m\} \times [s,t]) := r_m(t-s)$ $(m \in \mathcal{G},\ s \leq t)$ and let ω be a Poisson point set with intensity measure ρ. It is known [Swa22, Theorem 4.19] that almost surely, for each $x \in S^\Lambda$ and $s \in \mathbb{R}$, there exists a unique càdlàg function $[s,\infty) \ni t \mapsto X_t \in S^\Lambda$ that solves the evolution equation

$$X_s = x \quad \text{and} \quad X_t = \begin{cases} m(X_{t-}) & \text{if } (m,t) \in \omega, \\ X_{t-} & \text{else,} \end{cases} \qquad (t > s), \tag{1.5}$$

where $X_{t-} = \lim_{t'\uparrow t} X_{t'}$ denotes the state of the process just before time t. It is easy to see that almost surely, at any given time t, there can be at most one $m \in \mathcal{G}$ such that $(m,t) \in \omega$, so (1.5) is (a.s.) well-defined. We use the solutions of (1.5) to define random maps $\mathbf{X}_{s,u} : S^\Lambda \to S^\Lambda$ $(s \leq u)$ by

$$\mathbf{X}_{s,u}(x) := X_u, \quad \text{where } (X_u)_{u\geq s} \text{ solves (1.5)}. \tag{1.6}$$

These random maps form a *stochastic flow*, in the sense that $\mathbf{X}_{s,s}$ is the identity map for all $s \in \mathbb{R}$, and $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ $(s \le t \le u)$. Moreover, by [Swa22, Theorem 4.20], if X_0 is a random variable with values in S^Λ that is independent of the graphical representation ω, then setting

$$X_t := \mathbf{X}_{s,s+t}(X_0) \qquad (t \ge 0) \tag{1.7}$$

defines a Feller process $(X_t)_{t\ge 0}$ with càdlàg sample paths. By [Swa22, Theorem 4.30], the semigroup of this Feller process is the Feller semigroup with generator given in (1.4). We call this the interacting particle system with generator G.

1.3 Monotone systems duality

As in the previous subsection let S be a finite set and let Λ be countable. We assume from now on that S is equipped with a partial order $\le$ and has a least element 0, i.e. $0 \le a$ for all $a \in S$. For $x \in S^\Lambda$ we call $\mathrm{supp}(x) := \{i \in \Lambda : x(i) \neq 0\}$ the *support* of x and define

$$S^\Lambda_{\mathrm{fin}} := \big\{x \in S^\Lambda : \mathrm{supp}(x) \text{ is finite}\big\}. \tag{1.8}$$

For any $a \in S$, we let $\underline{a}$, defined as $\underline{a}(i) := a$ $(i \in \Lambda)$, denote the configuration that is constantly equal to a. The following lemma says that under natural assumptions, the space S^Λ_{fin} is preserved under the evolution of an interacting particle system.

Lemma 1 (Finite initial states) *Assume that*

$$m(\underline{0}) = m(\underline{0}) \qquad (m \in \mathcal{G}), \tag{1.9}$$

and that in addition to (1.3), the rates satisfy

$$\sup_{i\in\Lambda} \sum_{m\in\mathcal{G}} r_m \sum_{j\in\Lambda} 1_{\mathcal{D}(m)}(j) 1_{\mathcal{R}_2(m)}(i,j) < \infty. \tag{1.10}$$

Then one has almost surely

$$\mathbf{X}_{s,u}(x) \in S^\Lambda_{\mathrm{fin}} \qquad (s \le u,\ x \in S^\Lambda_{\mathrm{fin}}). \tag{1.11}$$

We equip S^Λ with the product order and we write $x < y$ if $x \le y$ and $x \neq y$ $(x, y \in S^\Lambda)$. For any set $A \subset S^\Lambda$, we call

$$A^\uparrow := \big\{x \in S^\Lambda : \exists y \in A \text{ s.t. } y \le x\big\} \tag{1.12}$$

the *upset* of A, and we say that A is *increasing* if $A^\uparrow = A$. We recall that a *minimal element* of a set $A \subset S^\Lambda$ is a configuration $x \in A$ such that there do not exist $y \in A$ with $y < x$. We let

$$A^\circ := \big\{x : x \text{ is a minimal element of } A\big\} \tag{1.13}$$

denote the set of minimal elements of A. Recall that S^Λ is equipped with the product topology. We set

$$\begin{aligned} \mathcal{I}(S^\Lambda) &:= \big\{A \subset S^\Lambda : A \text{ is open and increasing}\big\}, \\ \mathcal{H}(S^\Lambda) &:= \big\{Y \subset S^\Lambda_{\mathrm{fin}} : Y^\circ = Y\big\}. \end{aligned} \tag{1.14}$$

The following proposition describes a bijection between $\mathcal{I}(S^\Lambda)$ and $\mathcal{H}(S^\Lambda)$.

Proposition 2 (Encoding open increasing sets) *The map $Y \mapsto Y^\uparrow$ is a bijection from $\mathcal{H}(S^\Lambda)$ to $\mathcal{I}(S^\Lambda)$ and the map $A \mapsto A^\circ$ is its inverse.*

We will use the space

$$\mathcal{H}_-(S^\Lambda) := \mathcal{H}(S^\Lambda) \setminus \{\{\underline{0}\}\} \tag{1.15}$$

as the state space of the dual process. For that aim, we need to equip $\mathcal{H}_-(S^\Lambda)$ with a topology. We first equip $\mathcal{I}(S^\Lambda)$ with a topology and then use the bijection from Proposition 2 to transfer it to $\mathcal{H}(S^\Lambda)$. We will use the topology described in the following proposition. Note that $\{\underline{0}\}^\uparrow = S^\Lambda$ so that the space $\mathcal{I}_-(S^\Lambda)$ defined below corresponds to $\mathcal{H}_-(S^\Lambda)$ via the bijection of Proposition 2.

Proposition 3 (Dual topology) *There exists a unique metrizable topology on $\mathcal{I}(S^\Lambda)$ such that a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{I}(S^\Lambda)$ converges to a limit $A \in \mathcal{I}(S^\Lambda)$ if and only if*

$$1_{A_n}(x) \to 1_A(x) \quad \textit{for all} \quad x \in S^\Lambda_{\rm fin}. \tag{1.16}$$

The space $\mathcal{I}(S^\Lambda)$ is compact in this topology, and so is $\mathcal{I}_-(S^\Lambda) := \mathcal{I}(S^\Lambda) \setminus \{S^\Lambda\}$.

We equip $\mathcal{H}(S^\Lambda)$ with a topology so that the bijection from Proposition 2 is a homeomorphism. Then both $\mathcal{H}(S^\Lambda)$ and $\mathcal{H}_-(S^\Lambda)$ are compact metrizable spaces and a sequence $(Y_n)_{n\in\mathbb{N}} \subset \mathcal{H}(S^\Lambda)$ converges to a limit $Y \in \mathcal{H}(S^\Lambda)$ if and only if $1_{Y_n^\uparrow}(x) \to 1_{Y^\uparrow}(x)$ for all $x \in S^\Lambda_{\rm fin}$.

Let G be the generator of an interacting particle system with local state space S and grid Λ, written in the form (1.4), and assume that all maps $m \in \mathcal{G}$ are *monotone*, in the sense that

$$x \leq y \quad \text{implies} \quad m(x) \leq m(y) \qquad (x, y \in S^\Lambda). \tag{1.17}$$

Let $(\mathbf{X}_{s,u})_{s\leq u}$ be the stochastic flow defined in (1.6) in terms of the graphical representation ω.

Lemma 4 (Dual flow) *Assume that every map $m \in \mathcal{G}$ is monotone and satisfies (1.9), and that the rates satisfy (1.3). Then, almost surely, setting*

$$\mathbf{Y}_{u,s}(Y) := \big\{y \in S^\Lambda : \mathbf{X}_{s,u}(y) \in Y^\uparrow\big\}^\circ \qquad (Y \in \mathcal{H}(S^\Lambda),\ u \geq s) \tag{1.18}$$

yields a well-defined map $\mathbf{Y}_{u,s} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ for all $u \geq s$.

Note that $\mathbf{Y}_{u,s}(Y)$ is the collection of minimal configurations y with the property that if we start the interacting particle system at time s in the initial state y and evolve it under the graphical representation, then at the final time u the state of the interacting particle system lies in $Y^\uparrow$. Our first main result says that under natural assumptions, for fixed u and Y, setting $Y_t := \mathbf{Y}_{u,u-t}(Y)$ defines a Feller process with state space $\mathcal{H}_-(S^\Lambda)$. It is a consequence of our construction that this Feller process has, somewhat unusually, càglàd sample paths, i.e., its sample paths are left-continuous with right limits. In the upcoming theorem $\mathbb{P}$ denotes the probability measure on the probability space where the Poison point set ω lives.

Theorem 5 (Dual stochastic flow and Markov process) *Assume that every map $m \in \mathcal{G}$ is monotone and satisfies (1.9), and that the rates satisfy (1.3) and (1.10). Then, almost surely, (1.18) defines a map $\mathbf{Y}_{u,s} : \mathcal{H}_-(S^\Lambda) \to \mathcal{H}_-(S^\Lambda)$ for all $u \geq s$, and setting*

$$Q_t(Y, \cdot\,) := \mathbb{P}\big[\mathbf{Y}_{t,0}(Y) \in \cdot\,\big] \qquad (Y \in \mathcal{H}_-(S^\Lambda),\ t \geq 0) \tag{1.19}$$

defines a Feller semigroup on $\mathcal{H}_-(S^\Lambda)$. If $u \in \mathbb{R}$ and Y_0 is a random variable with values in $\mathcal{H}_-(S^\Lambda)$ that is independent of the graphical representation ω, then setting

$$Y_t := \mathbf{Y}_{u,u-t}(Y_0) \qquad (t \geq 0) \tag{1.20}$$

defines a Feller process $(Y_t)_{t\geq 0}$ with càglàd sample paths whose transition probabilities are $(Q_t)_{t\geq 0}$.

To compare this construction with earlier work, let

$$\mathcal{H}_{\rm fin}(S^\Lambda) := \big\{Y \in \mathcal{H}(S^\Lambda) : |Y| < \infty\big\} \tag{1.21}$$

denote the subset of $\mathcal{H}(S^\Lambda)$ consisting of the *finite* subsets $Y \subset S^\Lambda$ with $Y^\circ = Y$. Note that in this case we do not remove the element $\{\underline{0}\}$, and that $\mathcal{H}_{\rm fin}(S^\Lambda)$ is countable. We equip $\mathcal{H}_{\rm fin}(S^\Lambda)$ with the discrete topology. The following proposition says that under weaker assumptions than those of Theorem 5, for fixed $u \in \mathbb{R}$ and $Y \in \mathcal{H}_{\rm fin}(S^\Lambda)$, setting $Y_t := \mathbf{Y}_{u,u-t}(Y)$ defines a continuous-time Markov chain with countable state space $\mathcal{H}_{\rm fin}(S^\Lambda)$.

Proposition 6 (Finite initial states) *Assume that every map $m \in \mathcal{G}$ is monotone and the rates satisfy (1.3). Then, almost surely, (1.18) defines a map $\mathbf{Y}_{u,s} : \mathcal{H}_{\rm fin}(S^\Lambda) \to \mathcal{H}_{\rm fin}(S^\Lambda)$ for all $u \geq s$. If $u \in \mathbb{R}$ and Y_0 is a random variable with values in $\mathcal{H}_{\rm fin}(S^\Lambda)$ that is independent of the graphical representation ω, then setting*

$$Y_t := \mathbf{Y}_{u,u-t}(Y_0) \qquad (t \geq 0) \tag{1.22}$$

defines a continuous-time Markov chain $(Y_t)_{t\geq 0}$ in $\mathcal{H}_{\rm fin}(S^\Lambda)$ with càglàd sample paths.

In previous work [Gra86, SS18], the dual process $(Y_t)_{t\geq 0}$ has only been constructed for initial states $Y_0 \in \mathcal{H}_{\rm fin}(S^\Lambda)$. This more limited construction does not allow one to discuss the upper invariant law of the dual process, as we will do in the next subsection.

Let $\boldsymbol{\psi}_{\rm mon} : S^\Lambda \times \mathcal{H}(S^\Lambda) \to \{0,1\}$ be defined as

$$\boldsymbol{\psi}_{\rm mon}(x, Y) := 1_{Y\uparrow}(x) \qquad (x \in S^\Lambda,\ Y \in \mathcal{H}(S^\Lambda)). \tag{1.23}$$

Then using Proposition 2 and (1.18), it is straightforward to check that

$$\boldsymbol{\psi}_{\rm mon}\big(\mathbf{X}_{s,u}(x), Y\big) = \boldsymbol{\psi}_{\rm mon}\big(x, \mathbf{Y}_{u,s}(Y)\big) \qquad (s \leq u,\ x \in S^\Lambda,\ Y \in \mathcal{H}(S^\Lambda)). \tag{1.24}$$

We describe this in words by saying that the stochastic flow $(\mathbf{X}_{s,u})_{s\leq u}$ is dual to the backward stochastic flow $(\mathbf{Y}_{u,s})_{u\geq s}$ with respect to the *duality function* $\boldsymbol{\psi}_{\rm mon}$. Fix $s < u$ and let X_0 and Y_0 be random variables with values in S^Λ and $\mathcal{H}_-(S^\Lambda)$, respectively, independent of each other and of the graphical representation ω. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be the Markov processes defined by (1.7) and (1.20), i.e.

$$X_t := \mathbf{X}_{s,s+t}(X_0) \quad \text{and} \quad Y_t := \mathbf{Y}_{u,u-t}(Y_0) \qquad (t \geq 0). \tag{1.25}$$

Then setting $T := u - s$, (1.24) implies that

$$\boldsymbol{\psi}_{\rm mon}(X_t, Y_{T-t}) \quad \text{does not depend on } t \in [0,T]. \tag{1.26}$$

In particular, applying this to $t = 0$ and $t = T$ and taking expectations, it follows that the interacting particle system $(X_t)_{t\geq 0}$ with generator G in (1.4) and the Markov process $(Y_t)_{t\geq 0}$ from Theorem 5 are dual in the sense that

$$\mathbb{E}[\boldsymbol{\psi}_{\rm mon}(X_T, Y_0)] = \mathbb{E}[\boldsymbol{\psi}_{\rm mon}(X_0, Y_T)] \qquad (T \geq 0) \tag{1.27}$$

whenever X_T is independent of Y_0 and X_0 is independent of Y_T. Here $\mathbb{E}$ denotes expectation with respect to the probability measure on the probability space where ω, X_0 and Y_0 live. The relation (1.26) is called a *pathwise duality* relation and (1.27) means that the Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are *dual* with duality function $\boldsymbol{\psi}_{\rm mon}$.

The definition of the dual process in terms of the stochastic flow of the forward process as in (1.18) is somewhat indirect. In the remainder of this subsection, we will describe a more direct construction based on the graphical representation. We say that two functions $m : S^\Lambda \to S^\Lambda$ and $\hat{m} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ are *dual* with respect to the duality function $\boldsymbol{\psi}_{\rm mon}$ if

$$\boldsymbol{\psi}_{\rm mon}(m(x), Y) = \boldsymbol{\psi}_{\rm mon}(x, \hat{m}(Y)) \qquad (x \in S^\Lambda,\ Y \in \mathcal{H}(S^\Lambda)). \tag{1.28}$$

We need the following simple lemma.

Lemma 7 (Duals of monotone maps) *Each continuous monotone map* $m : S^\Lambda \to S^\Lambda$ *has a unique dual map* $\hat{m} : \mathcal{H}(S^\Lambda) \to \mathcal{H}(S^\Lambda)$ *with respect to the duality function* $\boldsymbol{\psi}_{\rm mon}$. *This dual map is given by*

$$\hat{m}(Y) := m^{-1}(Y^\uparrow)^\circ \qquad (Y \in \mathcal{H}(S^\Lambda)). \tag{1.29}$$

If m *satisfies (1.9), then moreover,* $\hat{m}(Y) \in \mathcal{H}_-(S^\Lambda)$ *for all* $Y \in \mathcal{H}_-(S^\Lambda)$.

The following proposition says that the backward stochastic flow $(\mathbf{Y}_{u,s})_{u\geq s}$ from (1.18) can be defined more directly in terms of the unique solutions of an evolution equation, similar to the definition of the forward stochastic flow $(\mathbf{X}_{s,u})_{s\geq u}$ in (1.5) and (1.6).

Proposition 8 (Backward evolution equation) *Under the assumptions of Theorem 5, almost surely, for each* $u \in \mathbb{R}$ *and* $Y \in \mathcal{H}_-(S^\Lambda)$, *there exists a unique càdlàg function* $(-\infty, u] \ni t \mapsto Y_t \in \mathcal{H}_-(S^\Lambda)$ *that solves the evolution equation*

$$Y_u = Y \quad \textit{and} \quad Y_{t-} = \begin{cases} \hat{m}(Y_t) & \textit{if } (m,t) \in \omega, \\ Y_t & \textit{else}, \end{cases} \qquad (t \leq u). \tag{1.30}$$

This function is given by $Y_t = \mathbf{Y}_{u,t}(Y)$ $(t \leq u)$, *where* $(\mathbf{Y}_{u,s})_{u\geq s}$ *is the backward stochastic flow defined in (1.18).*

We leave the problem of giving a generator characterization of the dual process to future work.

1.4 The upper invariant laws

Let E be a compact metrizable space that is equipped with a partial order $\leq$ that is *compatible with the topology*[4] in the sense that the set

$$\big\{(x,y) \in E^2 : x \leq y\big\} \quad \text{is a closed subset of } E^2, \tag{1.31}$$

where E^2 is equipped with the product topology. Two probability measure $\mu, \nu \in \mathcal{M}_1(E)$ are said to be *stochastically ordered*, denoted $\mu \leq \nu$, if they satisfy the following equivalent conditions [Lig85, Theorem II.2.4]

(i) $\int f(x)\,\mathrm{d}\nu(x) \leq \int f(x)\,\mathrm{d}\mu(x)$ for all continuous monotone $f : E \to \mathbb{R}$.

(ii) It is possible to couple random variables X, Y with laws μ, ν such that $X \leq Y$ a.s.

It is known [KK78, Theorem 2] that the stochastic order is a partial order on $\mathcal{M}_1(E)$. A Feller process with state space E and Feller semigroup $(P_t)_{t\geq 0}$ is said to be *monotone* if

$$P_t(x, \cdot\,) \leq P_t(y, \cdot\,) \qquad (x, y \in E,\ x \leq y). \tag{1.32}$$

The following result is well-known. It is stated for $E = \{0,1\}^\Lambda$ in [Lig85, Theorem III.2.3] and [Swa22, Theorem 5.4]. Generalizing the proof to all compact metrizable spaces equipped with a compatible topology is straightforward. The measure $\overline{\nu}$ below is called the *upper invariant law*.

[4]This notion is also used in [Lig85]. In the more classical references [Nac65] and [KK78] an order that satisfies (1.31) is called *closed*.

Proposition 9 (Upper invariant law) *Let E be a compact metrizable space equipped with a partial order that is compatible with the topology. Assume that E possesses a greatest element $\top \in E$, i.e. $x \leq \top$ for all $x \in E$. Let $(P_t)_{t\geq 0}$ be the semigroup of a monotone Feller process $(X_t)_{t\geq 0}$ with state space E. Then there exists an invariant law $\overline{\nu}$ of $(X_t)_{t\geq 0}$ that is uniquely characterized by the property that $\nu \leq \overline{\nu}$ for each invariant law ν of $(X_t)_{t\geq 0}$. Moreover, one has*

$$P_t(\top, \cdot\,) \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{1.33}$$

where $\Rightarrow$ denotes weak convergence of probability measures on E.

Returning to the set-up of the previous subsection, let S be a finite partially ordered set that has a least element 0 and let Λ be countable. Then it is easy to check that the product order on S^Λ is compatible with the topology. Let $(X_t)_{t\geq 0}$ be an interacting particle system with generator of the form (1.4). It follows from Lemma 21 below that if all maps $m \in \mathcal{G}$ are monotone, then (assuming (1.3)) the maps $\{\mathbf{X}_{s,u}\}_{s\leq u}$ are (a.s.) monotone for all $s \leq u$. This, in turn, implies that the interacting particle system $(X_t)_{t\geq 0}$ is monotone.[5] Thus, if the local state space S has a greatest element $\top$, then such an interacting particle system has an upper invariant law that is the long-time limit law started from the constant configuration $\underline{\top}$.

We equip the space $\mathcal{I}(S^\Lambda)$ with the partial order of set inclusion, which through the bijection of Proposition 2 defines a partial order $\leq$ on $\mathcal{H}(S^\Lambda)$ such that

$$Y \leq Z \quad \Leftrightarrow \quad Y^\uparrow \subset Z^\uparrow \qquad (Y, Z \in \mathcal{H}(S^\Lambda)). \tag{1.34}$$

Since $\{\underline{0}\}^\uparrow = S^\Lambda$, it is clear that $\{\underline{0}\}$ is the greatest element of $\mathcal{H}(S^\Lambda)$. It turns out that $\mathcal{H}_-(S^\Lambda)$ also has a greatest element, which is more interesting. If S is a partially ordered set that has a least element 0, then we set

$$S_{\rm sec} := (S \setminus \{0\})^\circ. \tag{1.35}$$

Elements of $S_{\rm sec}$ are "second from below" in the order on S. We define $Y_{\rm sec} \in \mathcal{H}(S^\Lambda)$ as

$$Y_{\rm sec} := \big\{\delta^a_i : i \in \Lambda,\ a \in S_{\rm sec}\big\} \tag{1.36}$$

where for any $a \in S$ and $i \in \Lambda$, we define $\delta^a_i \in S^\Lambda$ by

$$\delta^a_i(j) := \begin{cases} a & \text{if } j = i, \\ 0 & \text{else,} \end{cases} \qquad (j \in \Lambda). \tag{1.37}$$

The following lemma describes some elementary properties of the partial order on $\mathcal{H}(S^\Lambda)$.

Lemma 10 (Order on the dual state space) *The partial order $\leq$ defined in (1.34) is compatible with the topologies on $\mathcal{H}(S^\Lambda)$ and $\mathcal{H}_-(S^\Lambda)$, and $Y_{\rm sec}$ is the greatest element of $\mathcal{H}_-(S^\Lambda)$.*

It follows immediately from the definitions that the maps $\{\mathbf{Y}_{u,s}\}_{u\geq s}$ are monotone with respect to the partial order on $\mathcal{H}(S^\Lambda)$ for all $s \leq u$ as long as they are well defined. Hence the Feller process $(Y_t)_{t\geq 0}$ with state space $\mathcal{H}_-(S^\Lambda)$ defined in Theorem 5 is monotone. The abstract Proposition 9 therefore implies that this process has an upper invariant law $\overline{\mu}$ and that

$$Y_0 = Y_{\rm sec} \quad \text{implies that} \quad \mathbb{P}[Y_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\mu}. \tag{1.38}$$

[5] Remarkably, the converse statement does not hold. Having a generator of the form (1.4) with all maps monotone is strictly stronger than being monotone, see [FM01].

As far as we know, this upper invariant law has never been studied before, except in the special case when $(X_t)_{t\geq 0}$ is additive. Compare Section 1.5.

Recall the definitions of S^Λ_{fin} and $\mathcal{H}_{\text{fin}}(S^\Lambda)$ in (1.8) and (1.21). In view of Lemma 1 and Proposition 6, under the assumptions of Theorem 5, the Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, started in initial states in S^Λ_{fin} and $\mathcal{H}_{\text{fin}}(S^\Lambda)$, respectively, stay in these spaces for all times $t \geq 0$. We will relate the upper invariant laws of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ to the behavior of $(Y_t)_{t\geq 0}$ and $(X_t)_{t\geq 0}$ (in this order) started from finite initial states.

Let $\mathbb{P}$ again denote the probability measure on the probability space where the Poison point set ω lives. We say that the interacting particle system $(X_t)_{t\geq 0}$, respectively its monotone dual $(Y_t)_{t\geq 0}$ *dies out* if

$$\begin{aligned} &\mathbb{P}\big[\exists t \geq 0 \text{ s.t. } \mathbf{X}_{0,t}(x) = \underline{0}\big] = 1 \qquad (x \in S^\Lambda_{\text{fin}}),\\ &\mathbb{P}\big[\exists t \geq 0 \text{ s.t. } \mathbf{Y}_{t,0}(Y) = \emptyset\big] = 1 \qquad (Y \in \mathcal{H}_{\text{fin}}(S^\Lambda)). \end{aligned} \tag{1.39}$$

If these probabilities are less than one for some $x \in S^\Lambda_{\text{fin}}$ or $Y \in \mathcal{H}_{\text{fin}}(S^\Lambda)$, then we say that $(X_t)_{t\geq 0}$ or $(Y_t)_{t\geq 0}$ *survives.* The following proposition is a simple consequence of duality. Similar results have been exploited to great length for additive interacting particle systems.

Proposition 11 (Upper invariant law of the particle system) *Assume that every map $m \in \mathcal{G}$ is monotone, the rates satisfy (1.3), and that S has a greatest element $\top$. Let $\overline{X}$ be a random variable whose law is $\overline{\nu}$, the upper invariant law of the interacting particle system $(X_t)_{t\geq 0}$. Then $\overline{X} = \underline{0}$ a.s. if the dual process $(Y_t)_{t\geq 0}$ dies out and $\overline{X} \neq \underline{0}$ a.s. if the dual process $(Y_t)_{t\geq 0}$ survives.*

Thanks to the fact that we have constructed the dual process also for infinite initial states and have shown that it has an upper invariant law, we can now formulate an analogue result with the roles of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ reversed. At this point, the proof is not hard anymore. But, since this depends on all that has been done before, it is our second main result, and we formulate it as a theorem.

Theorem 12 (Upper invariant law of the dual process) *Assume that every map $m \in \mathcal{G}$ is monotone and satisfies (1.9), and that the rates satisfy (1.3) and (1.10). Let $\overline{Y}$ be a random variable whose law is $\overline{\mu}$, the upper invariant law of the dual process $(Y_t)_{t\geq 0}$. Then $\overline{Y} = \emptyset$ a.s. if the interacting particle system $(X_t)_{t\geq 0}$ dies out and $\overline{Y} \neq \emptyset$ a.s. if the interacting particle system $(X_t)_{t\geq 0}$ survives.*

1.5 Additive duality

Additive duality for interacting particle systems has been much studied and has found many applications since the foundational work of Harris [Har76, Har78] and Griffeath [Gri79]. Most work has been concerned with the local state space $S = \{0, 1\}$ but in [SS18] this has been generalized to S being a finite lattice. In fact, without using the terminology of lattice theory, additive duality for general finite sets S was already studied in [Fox16]. We will show that additive duality is a special case of monotone duality. We first need a few definitions.

Let S be a partially ordered set. In analogy with (1.12), the *downset* of a subset $A \subset S$ is defined as

$$A^\downarrow = \big\{b \in S : \exists a \in A \text{ s.t. } b \leq a\big\}, \tag{1.40}$$

and one says that A is *decreasing* if $A^\downarrow = A$. By definition, the *dual* of a partially ordered set S is a partially ordered set $\hat{S}$ together with a bijection $S \ni a \mapsto \hat{a} \in \hat{S}$ such that

$$a \leq b \quad \text{if and only if} \quad \hat{a} \geq \hat{b}. \tag{1.41}$$

All duals of a partially ordered set are naturally isomorphic and S is naturally isomorphic to the dual of $\hat{S}$ when one sets $\hat{\hat{a}} := a$. Note that if S has a least element 0, then $\hat{0}$ is the greatest element of $\hat{S}$ and vice versa. If S is a finite partially ordered set, Λ is countable, and S^Λ is equipped with the product order, then we define $\hat{x}(i) := \widehat{x(i)}$ $(x \in S^\Lambda,\ i \in \Lambda)$ coordinatewise. Then naturally $\hat{S}^\Lambda$ is dual to S^Λ.

A partially ordered set S is called a *lattice* if for every $a, b \in S$ there exist (necessarily unique) elements $a \vee b$ (the *join*) and $a \wedge b$ (the *meet*) such that

$$\{a\}^\uparrow \cap \{b\}^\uparrow = \{a \vee b\}^\uparrow \quad \text{and} \quad \{a\}^\downarrow \cap \{b\}^\downarrow = \{a \wedge b\}^\downarrow. \tag{1.42}$$

It is easy to see that each finite lattice S has a least element 0 and a greatest element $\top$. If S is a lattice, then so is S^Λ equipped with the product order, where $x \vee y$ and $x \wedge y$ are the coordinatewise join and meet of x and y.

We assume from now on that S is a finite lattice and Λ is a countable set. We let $\hat{S}$ denote the dual of S and we define a function $\boldsymbol{\psi}_{\text{add}} : S^\Lambda \times \hat{S}^\Lambda \to \{0, 1\}$ by

$$\boldsymbol{\psi}_{\text{add}}(x, \hat{y}) := 1_{(\{y\}^\downarrow)^{\text{c}}}(x) \qquad (x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda), \tag{1.43}$$

where $A^{\text{c}} := S^\Lambda \setminus A$ denotes the complement of a subset $A \subset S^\Lambda$. Put differently,

$$\boldsymbol{\psi}_{\text{add}}(x, \hat{y}) = \begin{cases} 0 & \text{if } x \leq y \text{ or equivalently } \hat{y} \leq \hat{x}, \\ 1 & \text{else,} \end{cases} \qquad (x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda). \tag{1.44}$$

In analogy with (1.28), we say that two functions $m : S^\Lambda \to S^\Lambda$ and $\hat{m} : \hat{S}^\Lambda \to \hat{S}^\Lambda$ are *dual* with respect to the duality function $\boldsymbol{\psi}_{\text{add}}$ if

$$\boldsymbol{\psi}_{\text{add}}(m(x), \hat{y}) = \boldsymbol{\psi}_{\text{add}}(x, \hat{m}(\hat{y})) \qquad (x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda). \tag{1.45}$$

We say that a map $m : S^\Lambda \to S^\Lambda$ is *additive* if

$$m(\underline{0}) = \underline{0} \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \quad (x, y \in S^\Lambda). \tag{1.46}$$

Clearly, each additive map m is also monotone. The following lemma combines [LS23a, Proposition 2.1, Lemma 2.4 & Proposition 2.5].

Proposition 13 (Additive systems pathwise duality) *Assume that every map $m \in \mathcal{G}$ is additive and that the rates satisfy (1.3) and (1.10). Then there (a.s.) exists a stochastic flow $(\mathbf{Z}_{u,s})_{u \geq s}$, consisting of random maps from $\hat{S}^\Lambda$ to itself, satisfying the relation*

$$\boldsymbol{\psi}_{\text{add}}\big(\mathbf{X}_{s,u}(x), \hat{y}\big) = \boldsymbol{\psi}_{\text{add}}\big(x, \mathbf{Z}_{u,s}(\hat{y})\big) \qquad (s \leq u,\ x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda). \tag{1.47}$$

We will show that the backward stochastic flow $(\mathbf{Z}_{u,s})_{u \geq s}$ can, in fact, be identified with the backward stochastic flow we have already seen. A non-empty, decreasing subset $I \subset S^\Lambda$ is called an *ideal* if it is closed under taking the join, i.e. if $x \vee y \in I$ for all $x, y \in I$. A *principal ideal* is an ideal that has a greatest element. Let

$$\mathcal{H}_{\text{pi}}(S^\Lambda) := \big\{Y \in \mathcal{H}(S^\Lambda) : (Y^\uparrow)^{\text{c}} \ \text{ is a principal ideal}\big\} \tag{1.48}$$

Note that $\mathcal{H}_{\text{pi}}(S^\Lambda) \subset \mathcal{H}_-(S^\Lambda)$. The following proposition identifies the partially ordered set $\mathcal{H}_{\text{pi}}(S^\Lambda)$ with the dual lattice $\hat{S}^\Lambda$ and shows that in this identification, the monotone duality function from (1.23) reduces to the additive duality function from (1.43).

Proposition 14 (Isomorphism to the dual lattice) *The partially ordered topological space $\hat{S}^\Lambda$ is isomorphic to $\mathcal{H}_{\text{pi}}(S^\Lambda)$ via the monotone homeomorphism $\phi : \hat{S}^\Lambda \to \mathcal{H}_{\text{pi}}(S^\Lambda)$ defined as*

$$\phi(\hat{y}) := \big[(\{y\}^\downarrow)^{\text{c}}\big]^\circ \qquad (\hat{y} \in \hat{S}^\Lambda). \tag{1.49}$$

Moreover,

$$\boldsymbol{\psi}_{\text{add}}(x, \hat{y}) = \boldsymbol{\psi}_{\text{mon}}(x, \phi(\hat{y})) \qquad (x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda). \tag{1.50}$$

The subspace $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ and the function ϕ from (1.49) are rather abstract. In Section 5 below, for the important special case of distributive lattices, we give an alternative description of $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$. Moreover, for S being a totally ordered lattice we compute ϕ explicitly.

Our final result says that for additive interacting particle systems, the backward stochastic flow $(\mathbf{Y}_{u,s})_{u\geq s}$ defined in (1.18) preserves the space $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$.

Proposition 15 (Preserved subspace) *Assume (1.3) and (1.10), assume that S is a finite lattice, and assume that all maps $m \in \mathcal{G}$ are additive. Then almost surely*

$$\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_{\mathrm{pi}}(S^\Lambda) \qquad (u \geq s,\ Y \in \mathcal{H}_{\mathrm{pi}}(S^\Lambda)). \tag{1.51}$$

By grace of Proposition 14 and 15, we can then identify the restriction of $(\mathbf{Y}_{u,s})_{u\geq s}$ to $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ with the backward stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$ defined in (1.47). The duality formulas (1.24), (1.26), and (1.27) now immediately translate into analogue formulas with $\boldsymbol{\psi}_{\mathrm{mon}}$ replaced by $\boldsymbol{\psi}_{\mathrm{add}}$ and $(\mathbf{Y}_{u,s})_{u\geq s}$ by $(\mathbf{Z}_{u,s})_{u\geq s}$.

1.6 Outline

The rest of the paper is devoted to proofs. We recall that we cited Proposition 9 from the literature. The space $\mathcal{H}(S^\Lambda)$ is studied in Section 2, where Proposition 2, Proposition 3 and Lemma 10 are proved. In Section 3 we first prove Lemma 4 and Lemma 1 (in this order). Afterwards we construct the dual process and prove the main result Theorem 5. In the second half of Section 3 we prove Lemma 7, Proposition 8 and Proposition 6 (in this order). Section 4 deals with the upper invariant laws. There we prove Proposition 11 and Theorem 12. Finally, Section 5 considers the additive special case. There we prove Proposition 13, Proposition 14 and Proposition 15. Moreover, as advertised, we further study $\mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ and ϕ from (1.49).

2 The dual space

In this section we prove Proposition 2, Proposition 3 and Lemma 10. To do so we first introduce a useful bit of notation. Let $x \in S^\Lambda$ and $\Delta \subset \Lambda$. Then we define $x\lfloor_\Delta \in S^\Lambda$ as

$$x\lfloor_\Delta (i) := \begin{cases} x(i) & \text{if } i \in I, \\ 0 & \text{else,} \end{cases} \qquad (i \in \Lambda). \tag{2.1}$$

The fact that, contrary to x_Δ from Section 1.2, $x\lfloor_\Delta$ is an element of S^Λ will be useful in several upcoming proofs. We can already prove Proposition 2.

Proof of Proposition 2. First note that $Y^\uparrow$ is indeed open for $Y \in \mathcal{H}(S^\Lambda)$. If $Y = \emptyset$, then also $Y^\uparrow = \emptyset$. If $Y = \{y\}$ for some $y \in S^\Lambda_{\mathrm{fin}}$, then

$$Y^\uparrow = \{y\}^\uparrow = \big\{x \in S^\Lambda : y(i) \leq x(i) \text{ for } i \in \mathrm{supp}(y)\big\}. \tag{2.2}$$

By the definitions of the product topology and of the discrete topology on S, all finite dimensional cylinder sets are open and hence, since $\mathrm{supp}(y)$ is finite, $Y^\uparrow$ is open[6] in the product topology. If Y consists of more than one element, then we can write

$$Y^\uparrow = \bigcup_{y\in Y} \{y\}^\uparrow \tag{2.3}$$

so that $Y^\uparrow$ is open as a union of open sets.

[6] In fact, $Y^\uparrow = \{y\}^\uparrow$ is clopen (i.e. both closed and open).

It then suffices to show that $(Y^{\uparrow})^{\circ} = Y$ for $Y \in \mathcal{H}(S^{\Lambda})$, $A^{\circ} \subset S^{\Lambda}_{\text{fin}}$ for $A \in \mathcal{I}(S^{\Lambda})$ and $(A^{\circ})^{\uparrow} = A$ for $A \in \mathcal{I}(S^{\Lambda})$. The first assertion is clear. We will show the third assertion and the arguments along the way will imply the second one as well.

Let $A \in \mathcal{I}(S^{\Lambda})$. Then $A^{\circ} \subset A$ implies that $(A^{\circ})^{\uparrow} \subset A^{\uparrow} = A$. If $A = \emptyset$, then $\emptyset \subset (\emptyset^{\circ})^{\uparrow}$ trivially and there is nothing left to show. Hence, assume that $A \neq \emptyset$ and let $x \in A$. Let $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of Λ with the property that $\Delta_n \nearrow \Lambda$. Then $x\lfloor_{\Delta_n} \to x$ in the product topology. As A is open there exists an $N \in \mathbb{N}$ so that $x\lfloor_{\Delta_n} \in A$ for all $n \geq N$. As $x\lfloor_{\Delta_N} \in S^{\Lambda}_{\text{fin}}$ we can find $x' \in A^{\circ}$ such that $x' \leq x\lfloor_{\Delta_N} \leq x$, thus $x \in (A^{\circ})^{\uparrow}$. In particular, this shows that there cannot exist an $x \in A^{\circ} \cap (S^{\Lambda}_{\text{fin}})^{\text{c}}$. □

Next, we construct the topology on $\mathcal{I}(S^{\Lambda})$ that satisfies Proposition 3. As the set Λ is countable, we can find a bijection $\gamma : \Lambda \to \mathbb{N}$. Using this bijection we define

$$a_i := \frac{1}{3^{\gamma(i)}} \qquad (i \in \Lambda) \tag{2.4}$$

and define a metric d on S^{Λ} as

$$d(x,y) := \sum_{i\in\Lambda} a_i 1\!\!1_{\{x(i)\neq y(i)\}} \qquad (x, y \in S^{\Lambda}). \tag{2.5}$$

It is well-known that the metric d generates the product topology on S^{Λ}. Note that $d(x,y) \leq 1/2$ $(x, y \in S^{\Lambda})$ with equality if and only if $x(i) \neq y(i)$ for all $i \in \Lambda$. Moreover, $d(x,y) < 1/3^n$ $(x, y \in S^{\Lambda})$ implies that $d(x,y) \leq 1/(2 \cdot 3^n) = \sum_{k=n+1}^{\infty} 1/3^k$, so that the open ball

$$\mathcal{B}_{1/3^n}(x) := \left\{ y \in S^{\Lambda} : d(x,y) < \frac{1}{3^n} \right\} \qquad (x \in S^{\Lambda}) \tag{2.6}$$

is actually clopen, i.e. both closed and open.

Let $\mathcal{K}(S^{\Lambda})$ denote the space of all compact subsets of S^{Λ}. On $\mathcal{K}_+(S^{\Lambda}) := \mathcal{K}(S^{\Lambda}) \setminus \{\emptyset\}$ one defines the *Hausdorff metric* d_{H} as

$$d_{\text{H}}(A,B) := \max \left\{ \sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A) \right\} \qquad (A, B \in \mathcal{K}_+(S^{\Lambda})), \tag{2.7}$$

where d is the metric from (2.5) and (as usual)

$$d(x,B) := \inf_{y\in B} d(x,y) \qquad (x \in S^{\Lambda},\ B \subset S^{\Lambda}). \tag{2.8}$$

The corresponding topology on $(\mathcal{K}_+(S^{\Lambda}), d_{\text{H}})$ is called *Hausdorff topology* and it is well-known (see for example [SSS14, Lemma B.1]) that it only depends on the topology on S^{Λ}, i.e. the product topology, and not on the exact definition of the underlying metric. However, using the metric d_{H} based on the concrete metric d form (2.5) will be useful in the following. We extend the metric d_{H} to $\mathcal{K}(S^{\Lambda})$ by setting $d_{\text{H}}(\emptyset, A) := 1$ for all $A \in \mathcal{K}_+(S^{\Lambda})$ so that $\emptyset$ is an isolated point. By [SSS14, Lemma B.3] the space $\mathcal{K}(S^{\Lambda})$ is then compact since S^{Λ} is compact.

We want to identify $\mathcal{I}(S^{\Lambda})$ with a subspace of $\mathcal{K}(S^{\Lambda})$. The assertion $A_n^{\updownarrow} \to A^{\updownarrow}$ in the following lemma is to be understood to mean that both $A_n^{\uparrow} \to A^{\uparrow}$ and $A_n^{\downarrow} \to A^{\downarrow}$.

Lemma 16 (Convergence of up- and downset) *Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{K}(S^{\Lambda})$ and assume that $A_n \to A \in \mathcal{K}(S^{\Lambda})$. Then also $A_n^{\updownarrow} \to A^{\updownarrow}$ in $\mathcal{K}(S^{\Lambda})$.*

For the proof of Lemma 16 we will need a classical result that is a special case of [Nac65, Proposition 4].

Lemma 17 (Closedness of upset and downset) *Let E be a compact Hausdorff space that is equipped with a partial order $\leq$ that is compatible with the topology. Assume that $A \subset E$ is closed. Then $A^{\uparrow}$ and $A^{\downarrow}$ are also closed.*

Proof of Lemma 16. Note that every $A \in \mathcal{K}(S^{\Lambda})$ is closed (in S^{Λ}), so by Lemma 17 the set $A^{\updownarrow}$ is closed as well. Hence indeed $A^{\updownarrow} \in \mathcal{K}(S^{\Lambda})$, as closed subsets of a compact topological space are compact.

Now assume that $A = \emptyset$. As $\emptyset$ is isolated in $\mathcal{K}(S^{\Lambda})$, $A_n \to A$ implies that there exists an $N \in \mathbb{N}$ so that $A_n = \emptyset$ for all $n \geq N$. As $\emptyset^{\uparrow} = \emptyset^{\downarrow} = \emptyset$ it follows that also $A_n^{\updownarrow} \to A^{\updownarrow}$.

Let now $A, B \in \mathcal{K}_{+}(S^{\Lambda})$ and $n \in \mathbb{N}$. We show that $d_{\mathrm{H}}(A, B) < 1/3^n$ implies that $d_{\mathrm{H}}(A^{\uparrow}, B^{\uparrow}) < 1/3^n$. Let $x \in B^{\uparrow}$ and $b \in B$ so that $b \leq x$. Then, as $d_{\mathrm{H}}(A, B) < 1/3^n$, there exists $a \in A$ such that $d(a, b) < 1/3^n$ which implies that $a_{\gamma^{-1}(\{1,\ldots,n\})} = b_{\gamma^{-1}(\{1,\ldots,n\})}$. Let now $a^x \in S^{\Lambda}$ be defined as

$$a^x(i) := \begin{cases} x(i) & \text{if } i \in \gamma^{-1}(\{1, \ldots, n\}), \\ a(i) & \text{else,} \end{cases} \qquad (i \in \Lambda). \tag{2.9}$$

Then $b \leq x$ implies that $a \leq a^x$ and the construction of a^x implies that $d(x, a^x) \leq 1/(2 \cdot 3^n)$ and hence $d(x, A^{\uparrow}) \leq 1/(2 \cdot 3^n)$. As $x \in B^{\uparrow}$ was arbitrary we conclude that $\sup_{x \in B^{\uparrow}} d(x, A^{\uparrow}) \leq 1/(2 \cdot 3^n)$. Interchanging the roles of B and A yields that $d_{\mathrm{H}}(B^{\uparrow}, A^{\uparrow}) \leq 1/(2 \cdot 3^n) < 1/3^n$. The argument for $\uparrow$ replaced by $\downarrow$ works analogously. □

Let $A \in \mathcal{I}(S^{\Lambda})$. Then A^{c}, being a closed subset of a compact topological space, is compact. We have the following.

Lemma 18 (Closedness within compact sets) *The set $\{A^{\mathrm{c}} : A \in \mathcal{I}(S^{\Lambda})\}$ is closed in $\mathcal{K}(S^{\Lambda})$.*

Proof. Let $(A_n)_{n \in \mathbb{N}} \subset \{A^{\mathrm{c}} : A \in \mathcal{I}(S^{\Lambda})\}$ and assume that $A_n \to A \in \mathcal{K}(S^{\Lambda})$. As each A_n ($n \in \mathbb{N}$) is decreasing, Lemma 16 shows that also $A_n \to A^{\downarrow}$. The Hausdorff property of $\mathcal{K}(S^{\Lambda})$ then implies that $A = A^{\downarrow}$ and the proof is complete. □

Now we are ready to prove Proposition 3.

Proof of Proposition 3. Using Lemma 18 we can equip $\mathcal{I}(S^{\Lambda})$ with the metric

$$d_{\mathcal{I}}(A, B) := d_{\mathrm{H}}(A^{\mathrm{c}}, B^{\mathrm{c}}) \qquad (A, B \in \mathcal{I}(S^{\Lambda})) \tag{2.10}$$

making $(\mathcal{I}(S^{\Lambda}), d_{\mathcal{I}})$ and $(\mathcal{I}_{-}(S^{\Lambda}), d_{\mathcal{I}})$ compact metric spaces, isometric to some closed subspaces of the metric space $(\mathcal{K}(S^{\Lambda}), d_{\mathrm{H}})$.

Next we prove the convergence criterion. To start we consider the case $A = S^{\Lambda}$. As S^{Λ} is isolated in $\mathcal{I}(S^{\Lambda})$, $A_n \to S^{\Lambda}$ implies that there exists $N \in \mathbb{N}$ such that $A_n = S^{\Lambda}$ for all $n \geq N$ so that (1.16) is trivial. On the other hand, assuming (1.16) and taking $x = \underline{0}$ implies that there has to exist an $N \in \mathbb{N}$ such that $\underline{0} \in A_n$ for all $n \geq N$. But $\underline{0} \in A_n$ implies $\underline{0} \in A_n^{\circ}$ and by minimality $A_n^{\circ} = \{\underline{0}\}$. Hence, by Proposition 2, $A_n = S^{\Lambda}$ for all $n \geq N$ so that $A_n \to S^{\Lambda}$.

Assume now that $A \in \mathcal{I}_{-}(S^{\Lambda})$. If $x = \underline{0}$ violates (1.16), then, by the arguments above, $A_n = S^{\Lambda}$ for infinitely many $n \in \mathbb{N}$ and A_n cannot converge to A. Now assume that there exists an $x \in S^{\Lambda}_{\mathrm{fin}} \setminus \{\underline{0}\}$ such that $\mathbb{1}_{A_n}(x)$ does not converge to $\mathbb{1}_A(x)$. This implies that for all $n \in \mathbb{N}$ there exists an $N \geq n$ such that $x \in A_N \triangle A = (A_N \setminus A) \cup (A \setminus A_N)$. Let now $a_* := \min\{a_i : i \in \mathrm{supp}(x)\}$ and $m := \log_{1/3}(a_*)$. We claim that $x \in A_N \triangle A$ implies that $d_{\mathrm{H}}(A_N^{\mathrm{c}}, A^{\mathrm{c}}) \geq a_*$ and hence also

$$\limsup_{n \to \infty} d_{\mathrm{H}}(A_n^{\mathrm{c}}, A^{\mathrm{c}}) \geq a_*. \tag{2.11}$$

To check the claim, due to symmetry we may w.l.o.g. assume that $x \in A_N \cap A^c$. Would there now be a $y \in A_N^c$ with $d(x,y) < a_* = 1/3^m$, then, as $\mathrm{supp}(x) \subset \gamma^{-1}(\{1,\dots,m\})$, we would have $y\lfloor_{\mathrm{supp}(x)} = x$. But, as A_N^c is decreasing, $x = y\lfloor_{\mathrm{supp}(x)} \leq y$ implies $x \in A_N^c$, a contradiction.

For the reverse direction let $\varepsilon > 0$. Choose an $m \in \mathbb{N}$ such that $1/(2 \cdot 3^m) < \varepsilon$. By assumption for all $x \in S^\Lambda_{\mathrm{fin}}$ there exists an $N(x) \in \mathbb{N}$ such that $\mathbb{1}_{A_n}(x) = \mathbb{1}_A(x)$ for all $n \geq N(x)$. Set now

$$N_0 := \max\left\{N(x) : \mathrm{supp}(x) \subset \gamma^{-1}(\{1,\dots,m\})\right\}. \tag{2.12}$$

We claim that this implies that $d_{\mathrm{H}}(A_n^c, A^c) < \varepsilon$ for all $n \geq N_0$ and hence

$$\limsup_{n\to\infty} d_{\mathrm{H}}(A_n^c, A^c) = 0, \tag{2.13}$$

as ε was arbitrary. To check the claim, let $n \geq N_0$ and assume that there exists an $x \in S^\Lambda$ with arbitrary support satisfying $x \in A_n^c \cap A$. We then also have that $x\lfloor_{\gamma^{-1}(\{1,\dots,m\})} \in A_n^c$ which implies that $x\lfloor_{\gamma^{-1}(\{1,\dots,m\})} \in A^c$ as $n \geq N_0$. Due to the construction of d this implies that $d(x, A^c) \leq 1/(2 \cdot 3^{N_0})$. By symmetry, an arbitrary $x \in A_n \cap A^c$ has to satisfy $d(x, A_n^c) \leq 1/(2 \cdot 3^{N_0})$ and the claim follows.

The uniqueness of the metrizable topology that satisfies (1.16) follows from the fact that convergence of sequences characterizes a topology in metrizable spaces. □

We finish the section with the proof of Lemma 10.

Proof of Lemma 10. Let $Y, Z \in \mathcal{H}(S^\Lambda)$. As

$$Y \leq Z \quad \Leftrightarrow \quad Y^\uparrow \subset Z^\uparrow \quad \Leftrightarrow \quad (Z^\uparrow)^c \subset (Y^\uparrow)^c \tag{2.14}$$

if suffices to show that $\mathcal{K}(S^\Lambda)$ satisfies (1.31) if it is equipped with the partial order $\subset$. Let $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{K}(S^\Lambda)$ be two sequences such that $A_n \subset B_n$ for all $n \in \mathbb{N}$ and assume that $A_n \to A \in \mathcal{K}_+(S^\Lambda)$ and $B_n \to B \in \mathcal{K}_+(S^\Lambda)$. If $A = \emptyset$, then trivially $A \subset B$. Hence, assume that $A \neq \emptyset$ and let $a \in A$. From the definition of the Hausdorff metric we can conclude that there exist $a_n \in A_n$ $(n \in \mathbb{N})$ so that $a_n \to a$ in S^Λ. But then $a_n \in B_n$ $(n \in \mathbb{N})$ and by [SSS14, Lemma B.1] this implies that $a \in B$. From this one concludes that $A \subset B$ and hence $\mathcal{K}(S^\Lambda)$ satisfies (1.31).

Finally, we consider Y_{sec}. Clearly, $\underline{0} \notin Y_{\mathrm{sec}}$, so $Y_{\mathrm{sec}} \in \mathcal{H}_-(S^\Lambda)$. Let $Y \in \mathcal{H}_-(S^\Lambda)$. Since $\underline{0} \notin Y$, for every $y \in Y$ there has to exist an $i \in \Lambda$ satisfying $y(i) \neq 0$. But then there also exists an $a \in S_{\mathrm{sec}}$ such that $a \leq y(i)$. Hence also $\delta_i^a \leq y$ and thus $y \in Y_{\mathrm{sec}}^\uparrow$. It follows that $Y^\uparrow \subset Y_{\mathrm{sec}}^\uparrow$, i.e. $Y \leq Y_{\mathrm{sec}}$. □

3 The dual process

We split this section into two subsections. In Subsection 3.1 we begin by proving Lemma 4 and Lemma 1. Afterwards we construct the dual process and prove the main result Theorem 5. Subsection 3.2 is devoted to duality between maps, where Lemma 7 is the cornerstone. With Lemma 7 at our disposal we also show Proposition 8 and Proposition 6 (in this order).

3.1 The Poisson construction

As outlined, we begin this subsection by proving Lemma 4. This needs some preparations. For a function $f : S^\Lambda \to S$ we set

$$\mathcal{R}(f) := \left\{i \in \Lambda : \exists x, y \in S^\Lambda \text{ s.t. } f(x) \neq f(x') \text{ and } x_{\Lambda\setminus\{i\}} = y_{\Lambda\setminus\{i\}}\right\}. \tag{3.1}$$

Moreover, for $m : S^\Lambda \to S^\Lambda$ and $i \in \Lambda$ we define $m[i] : S^\Lambda \to S$ via $m[i](x) := m(x)(i)$ ($x \in S^\Lambda$). Let $m : S^\Lambda \to S^\Lambda$ be a local map. With the newly introduced notation we can write $\mathcal{R}_2(m)$ from (1.2) as

$$\mathcal{R}_2(m) = \big\{(j,i) \in \Lambda^2 : i \in \Lambda,\ j \in \mathcal{R}(m[i])\big\}. \tag{3.2}$$

We cite the following result [Swa22, Lemma 4.13].

Lemma 19 (Continuous maps) *A map $f : S^\Lambda \to S$ is continuous with respect to the product topology if and only if the following two conditions hold:*

(i) *$\mathcal{R}(f)$ is finite.*

(ii) *If $x_1, x_2 \in S^\Lambda$ satisfy $x_1(j) = x_2(j)$ for all $j \in \mathcal{R}(f)$, then $f(x_1) = f(x_2)$.*

Recall that the Poisson set ω from (1.5) is called the graphical representation. For each $s \le u$, we set

$$\omega_{s,u} := \big\{(m,t) \in \omega : s < t \le u\big\}. \tag{3.3}$$

For each finite $\tilde\omega \subset \omega_{s,u}$, we define

$$\mathbf{X}^{\tilde\omega}_{s,u} := m_n \circ \cdots \circ m_1 \quad \text{with} \quad \tilde\omega = \big\{(m_1,t_1),\dots,(m_n,t_n)\big\} \quad \text{and} \quad t_1 < \cdots < t_n, \tag{3.4}$$

i.e., $\mathbf{X}^{\tilde\omega}_{s,u}$ is the concatenation of the maps from $\tilde\omega$ in the time order in which they occur. The following result follows from [Swa22, Lemma 4.24] and the proof of [Swa22, Theorem 4.19].

Lemma 20 (Finitely many relevant local maps) *Assume (1.3). Then almost surely, for each $s \le u$ and $i \in \Lambda$, there exist a finite sets $\omega_{s,u}(i) \subset \omega_{s,u}$ such that*

$$\mathbf{X}_{s,u}[i] = \mathbf{X}^{\tilde\omega}_{s,u}[i] \quad \textit{for all finite } \tilde\omega \textit{ with } \omega_{s,u}(i) \subset \tilde\omega \subset \omega_{s,u}. \tag{3.5}$$

These finite sets can be chosen such that $\omega_{t,u}(i) = \omega_{s,u}(i) \cap \omega_{t,u}$ for all $s \le t \le u$ and $i \in \Lambda$.

With the help of the previous two lemmas, we can prove the following result.

Lemma 21 (Continuous monotone flow) *Assuming (1.3) and that all maps $m \in \mathcal{G}$ are monotone, almost surely, the maps $\mathbf{X}_{s,u} : S^\Lambda \to S^\Lambda$ ($s \le u$), defined in (1.6), are continuous and monotone.*

Proof. By Lemma 20, $\mathbf{X}_{s,u}[i]$ is the concatenation of finitely many monotone maps and therefore monotone. As we equipped S^Λ with the product order, the same holds for $\mathbf{X}_{s,u}$.

Recall that we equipped S^Λ with the product topology. The definition of a local map in Section 1.2 together with Lemma 19 implies that m is continuous for all $m \in \mathcal{G}$. Then $\mathbf{X}_{s,u}[i]$, being the concatenation of finitely many continuous maps, is continuous. By the properties of the product topology $\mathbf{X}_{s,u}$ is then continuous as well. This concludes the proof. □

Before we continue with the proof of Lemma 4 we note the following consequence of Lemma 20.

Corollary 22 (The trap $\underline{0}$) *Assuming (1.3) and (1.9), almost surely $\mathbf{X}_{s,u}(\underline{0}) = \underline{0}$ for all $s \le u$.*

Proof of Lemma 4. Let $u \ge s$, $Y \in \mathcal{H}(S^\Lambda)$, and let $A := \{y \in S^\Lambda : \mathbf{X}_{s,u}(y) \in Y^\uparrow\}$ be the preimage of $Y^\uparrow$ under the map $\mathbf{X}_{s,u}$. By Lemma 21 $\mathbf{X}_{s,u}$ is almost surely continuous and monotone. The continuity of $\mathbf{X}_{s,u}$ implies that A is open and the montonicity of $\mathbf{X}_{s,u}$ implies that A is increasing. By (1.18) and Proposition 2, it now follows that $\mathbf{Y}_{u,s}(Y)^\uparrow = A$. Hence $\mathbf{Y}_{u,s}(Y)^\uparrow \in \mathcal{I}(S^\Lambda)$ and $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}(S^\Lambda)$ by Proposition 2. □

Next we prove Lemma 1. Although the result is widely applicable we were not able to find a reference for the result. This might be due to the fact that the assertion of Lemma 1 follows for many well-studied interacting particle systems from duality (compare [SS18, Lemma 32]). The proof is done in parallel to the proofs of [Swa22, Lemma 4.21 & Lemma 4.22]. Recall that $\mathrm{supp}(x)$ was introduced for $x \in S^\Lambda$ at the beginning of Section 1.3.

Proof of Lemma 1. Recall the definitions in (1.2) and (3.1). We start the proof with the following observation. Let $m \in \mathcal{G}$. If $j \in \mathrm{supp}(m(x))$, then there are two possibilities. Either $j \notin \mathcal{D}(m)$, i.e. $m(x)(j) = x(j)$, and hence $j \in \mathrm{supp}(x)$ has to hold as well. On the other hand, if $j \in \mathcal{D}(m)$, then there has to exist an $i \in \mathcal{R}(m[i]) \cap \mathrm{supp}(x)$. Indeed, by (1.9) and Lemma 19 (ii), $x(i) = 0$ for all $i \in \mathcal{R}(m[i])$ implies $m(x)(j) = m(\underline{0})(j) = 0$. This observation gives rise to the following construction. For $i, j \in \Lambda$ and $s \le u$ we write $(i, s) \rightsquigarrow (j, u)$ if there exists a càdlàg function $\xi : [s, u] \to \Lambda$ with $\xi(s) = i$, $\xi(u) = j$ and the property that

- if $\xi(t-) \neq \xi(t)$ for some $t \in (s, u]$, then there exists a map $m \in \mathcal{G}$ such that $(m, t) \in \omega$, $\xi(t) \in \mathcal{D}(m)$ and $\xi(t-) \in \mathcal{R}(m[\xi(t)])$.

We set

$$\zeta_{s,u}(I) := \big\{ j \in \Lambda : (i, s) \rightsquigarrow (j, u) \text{ for some } i \in I \big\} \qquad (s \le u,\ I \subset \Lambda). \tag{3.6}$$

Note that $\zeta_{s,s}(I) := I$ ($s \in \mathbb{R}$, $I \subset \Lambda$). Our earlier observations and the fact that $(\mathbf{X}_{s,u}(x))_{u \ge s}$ ($s \in \mathbb{R}$, $x \in S^\Lambda$) solves (1.5) then imply that

$$\mathrm{supp}(\mathbf{X}_{s,u}(x)) \subset \zeta_{s,u}(\mathrm{supp}(x)) \qquad (s \le u,\ x \in S^\Lambda). \tag{3.7}$$

Note that, by definition, we just require jumps of the càdàg function ξ to correspond to instances $(m, t) \in \omega$ while ξ may ignore some $(m, t) \in \omega$ "on its way". Hence, $I \subset \zeta_{s,u}(I)$ and

$$\zeta_{s,u}(I) \subset \zeta_{\lfloor s \rfloor, \lceil u \rceil}(I) \qquad (s \le u,\ I \subset \Lambda), \tag{3.8}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and the ceiling of a real number, respectively. Recall that S^Λ_{fin} is countable. Hence, by (3.7) and (3.8), in order to show that

$$|\mathrm{supp}(\mathbf{X}_{s,u}(x))| \tag{3.9}$$

is almost surely finite *for all* $x \in S^\Lambda_{\mathrm{fin}}$ and $s \le u$, it suffices to show that

$$|\zeta_{s,u}(\mathrm{supp}(x))| \tag{3.10}$$

is almost surely finite for *fixed* $x \in S^\Lambda_{\mathrm{fin}}$ and $s \le u$ with $s, u \in \mathbb{Z}$. To do so we use a standard generator computation. Let $(\Delta_n)_{n \in \mathbb{N}}$ be a sequence of finite subsets of Λ with $\Delta_n \nearrow \Lambda$ as $n \to \infty$. Let $\mathcal{P}(\Delta_n)$ denote the power set of Δ_n ($n \in \mathbb{N}$). We define

$$\zeta^n_{s,u}(I) := \big\{ j \in \Lambda : (i, s) \rightsquigarrow_n (j, u) \text{ for some } i \in I \big\} \qquad (n \in \mathbb{N},\ s \le u,\ I \subset \Delta_n), \tag{3.11}$$

where $(i, s) \rightsquigarrow_n (j, u)$ is defined as $(i, s) \rightsquigarrow (j, u)$ above, but with the càdlàg function ξ being required to map into Δ_n instead of the whole Λ. We claim that $(\zeta^n_{s,u}(\mathrm{supp}(x)))_{u \ge s}$ is a Markov process with finite state space $\mathcal{P}(\Delta_n)$ and generator G_n of the form

$$G_n f(I) := \sum_{m \in \mathcal{G}} r_m \{ f(m_n(I)) - f(I) \} \qquad (n \in \mathbb{N},\ I \subset \Delta_n), \tag{3.12}$$

where for all $m \in \mathcal{G}$ and $n \in \mathbb{N}$ we define $m_n : \mathcal{P}(\Delta_n) \to \mathcal{P}(\Delta_n)$ by

$$m_n(I) := I \cup \big[\{ j \in \mathcal{D}(m) : \exists i \in I \cap \mathcal{R}(m[j]) \} \cap \Delta_n \big] \qquad (n \in \mathbb{N},\ I \subset \Delta_n). \tag{3.13}$$

Indeed, this follows from standard theory (compare [Swa22, Theorem 2.7]) and the fact that

$$\sum_{\substack{m\in\mathcal{G}:\\ m_n(I)\neq I}} r_m \le \sum_{i\in\Delta_n}\sum_{m\in\mathcal{G}} r_m \sum_{j\in\Lambda} 1_{\mathcal{D}(m)}(j) 1_{\mathcal{R}_2}(i,j) < \infty \qquad (I\subset\Delta_n), \tag{3.14}$$

which says that the total rate of Poisson events that can change the state of the process is finite in any state $I \in \mathcal{P}(\Delta_n)$. Choosing f to be the function computing the cardinality of a set, i.e. $f(I) := |I|$, one has that

$$G_n f(I) \le \sum_{i\in I}\sum_{m\in\mathcal{G}} r_m \sum_{j\in\Lambda} 1_{\mathcal{D}(m)}(j) 1_{\mathcal{R}_2}(i,j) \le K f(I) \qquad (n\in\mathbb{N},\ I\subset\Delta_n), \tag{3.15}$$

where $K < \infty$ is the supremum in (1.10). Standard theory (compare the proof of [Swa22, Lemma 4.21]) then implies that

$$\mathbb{E}\big[|\zeta^n_{s,u}(\mathrm{supp}(x))|\big] \le |\mathrm{supp(x)}| e^{K(u-s)} < \infty \qquad (n\in\mathbb{N},\ u\in\mathbb{Z}: u\ge s). \tag{3.16}$$

Letting $n \to \infty$, using monotone convergence, it follows that $|\zeta_{s,u}(\mathrm{supp}(x)))|$ is almost surely finite. □

Now we deal with the proof of Theorem 5. We split it into smaller pieces. We begin with the following.

Lemma 23 (Avoiding the greatest element) *Assume that every map $m \in \mathcal{G}$ is monotone and satisfies (1.9), and that the rates satisfy (1.3). Then, almost surely,*

$$\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_-(S^\Lambda) \qquad (u\ge s,\ Y\in\mathcal{H}_-(S^\Lambda)). \tag{3.17}$$

Proof. Let $Y \in \mathcal{H}_-(S^\Lambda)$. Using (1.24), Corollary 22 and the definition of $\boldsymbol{\psi}_{\mathrm{mon}}$,

$$0 = \boldsymbol{\psi}_{\mathrm{mon}}(\underline{0}, Y) = \boldsymbol{\psi}_{\mathrm{mon}}\big(\mathbf{X}_{s,u}(\underline{0}), Y\big) = \boldsymbol{\psi}_{\mathrm{mon}}\big(\underline{0}, \mathbf{Y}_{u,s}(Y)\big), \tag{3.18}$$

thus $\mathbf{Y}_{u,s}(Y) \neq \{\underline{0}\}$, i.e. $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_-(S^\Lambda)$. □

Next we replace the assumptions (1.9) and (1.10) in Theorem 5 by different ones that make the proof easier.

Lemma 24 (Backward Feller process) *Assume that every map $m \in \mathcal{G}$ is monotone, that the rates satisfy (1.3) and*

$$\sum_{m\in\mathcal{G}:\, m(x)\neq x} r_m < \infty \quad \textit{for all } x \in S^\Lambda_{\mathrm{fin}}, \tag{3.19}$$

and that (1.11) almost surely holds. Then the conclusions in Theorem 5 hold.

Proof. That the first assertion of Theorem 5 holds was shown in Lemma 23. Moreover, by construction, the stochastic flow $(\mathbf{Y}_{u,s})_{u\ge s}$ has independent increments meaning that $\mathbf{Y}_{t_1,t_0}, \mathbf{Y}_{t_2,t_1}, \ldots, \mathbf{Y}_{t_n,t_{n-1}}$ are independent for all $t_0 < t_1 < \ldots < t_n$ $(n \in \mathbb{N})$ and $\mathbf{Y}_{u,s}$ and $\mathbf{Y}_{u+t,s+t}$ $(u \ge s,\ t \in \mathbb{R})$ are identically distributed. These two facts imply that $(Y_t)_{t\ge0}$ defined by (1.20) is a Markov process with Markov semigroup $(Q_t)_{t\ge0}$ (compare [Swa22, Proofs of Theorem 4.20 and Proposition 2.7]).

Hence to conclude that $(Y_t)_{t\ge0}$ is a Feller process it suffices to show that

$$(Y,t) \mapsto Q_t(Y,\,\cdot\,) \text{ is a continuous map from } \mathcal{H}_-(S^\Lambda)\times[0,\infty) \text{ to } \mathcal{M}_1(\mathcal{H}_-(S^\Lambda)). \tag{3.20}$$

Let $((Y_n, t_n))_{n\in\mathbb{N}} \subset \mathcal{H}_-(S^\Lambda) \times [0, \infty)$ such that $(Y_n, t_n) \to (Y, t) \in \mathcal{H}_-(S^\Lambda) \times [0, \infty)$ as $n \to \infty$ (where $\mathcal{H}_-(S^\Lambda)\times[0, \infty)$ is equipped with the product topology). Since almost sure convergence implies weak convergence in law it suffices to show that

$$\mathbf{Y}_{t_n,0}(Y_n) \underset{n\to\infty}{\longrightarrow} \mathbf{Y}_{t,0}(Y) \quad \text{a.s.} \tag{3.21}$$

By Proposition 3 we have to show that

$$\mathbb{1}_{(\mathbf{Y}_{t_n,0}(Y_n))^\uparrow}(x) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{(\mathbf{Y}_{t,0}(Y))^\uparrow}(x) \quad \text{a.s.} \tag{3.22}$$

for each $x \in S^\Lambda_{\text{fin}}$. By (1.24) this is equivalent to

$$\mathbb{1}_{Y_n^\uparrow}\big(\mathbf{X}_{0,t_n}(x)\big) \underset{n\to\infty}{\longrightarrow} \mathbb{1}_{Y^\uparrow}\big(\mathbf{X}_{0,t}(x)\big) \quad \text{a.s.} \tag{3.23}$$

for each $x \in S^\Lambda_{\text{fin}}$. Let

$$I(x) := \big\{u \in \mathbb{R} : \exists (m, u) \in \omega : m(x) \neq x\big\}. \tag{3.24}$$

Then, due to (3.19), $I(x)$ is a Poison point set on $\mathbb{R}$ with finite intensity. Let now

$$\begin{aligned} t_- &:= \sup\{u \in I(x) : u \leq t\}, \\ t_+ &:= \inf\{u \in I(x) : u \geq t\}. \end{aligned} \tag{3.25}$$

Since t is a deterministic time, $t_- < t < t_+$ a.s. Since $\mathbf{X}_{0,t_n}(x) = \mathbf{X}_{0,t}(x)$ for all n large enough so that $t_- < t_n < t_+$, (3.23) follows from Proposition 3 as $\mathbf{X}_{0,t}(x) \in S^\Lambda_{\text{fin}}$ a.s. by assumption.

Finally, we show that $(Y_t)_{t\geq 0}$ has (a.s.) càglàd sample paths. Fix $u \in \mathbb{R}$. We show that $(Y_t)_{t\geq 0}$ has (a.s.) càglàd sample paths by proving that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_-(S^\Lambda)$ has (a.s.) càdlàg sample paths for all $Y \in \mathcal{H}_-(S^\Lambda)$. As indicated above, (3.19) implies that (a.s.)

$$I_{s_1,s_2}(x) := I(x) \cap (s_1, s_2] \text{ is finite for all } s_1 \leq s_2 \text{ and } x \in S^\Lambda_{\text{fin}}. \tag{3.26}$$

For any $Y \in \mathcal{H}_-(S^\Lambda)$ and $t < u$ choose an arbitrary sequence $(s_n)_{n\in\mathbb{N}} \subset (t, u]$ with $s_n \searrow t$. We show that $\mathbf{Y}_{u,s_n}(Y) \to \mathbf{Y}_{u,t}(Y)$ in $\mathcal{H}_-(S^\Lambda)$ as $n \to \infty$. Using Proposition 3 and the definition of $\boldsymbol{\psi}_{\text{mon}}$ this is equivalent to showing that

$$\boldsymbol{\psi}_{\text{mon}}\big(x, \mathbf{Y}_{u,s_n}(Y)\big) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_{\text{mon}}\big(x, \mathbf{Y}_{u,t}(Y)\big) \quad \text{for all } x \in S^\Lambda_{\text{fin}}. \tag{3.27}$$

By (1.24) this again is equivalent to showing

$$\boldsymbol{\psi}_{\text{mon}}\big(\mathbf{X}_{s_n,u}(x), Y\big) \underset{n\to\infty}{\longrightarrow} \boldsymbol{\psi}_{\text{mon}}\big(\mathbf{X}_{t,u}(x), Y\big) \quad \text{for all } x \in S^\Lambda_{\text{fin}}. \tag{3.28}$$

By (3.26) for all $t \in \mathbb{R}$ and $x \in S^\Lambda_{\text{fin}}$ there (a.s.) exists an $\varepsilon > 0$ such that $I_{t,t+\varepsilon}(x) = \emptyset$. Due to (1.6) and (1.7), in this case $\mathbf{X}_{s_n,u}(x) = \mathbf{X}_{t,u}(x)$ for all $s_n \in (t, t + \varepsilon]$ and (3.28) follows. Thus, $t \mapsto \mathbf{Y}_{u,t}(Y)$ is (a.s.) right-continuous.

For any $Y \in \mathcal{H}_-(S^\Lambda)$ and $t \leq u$ choose an arbitrary sequence $(s_n)_{n\in\mathbb{N}} \subset (-\infty, t)$ with $s_n \nearrow t$. We show that $\mathbf{Y}_{u,s_n}(Y)$ has a limit as $n \to \infty$. With the arguments from above we can equivalently show that $\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s_n,u}(x), Y)$ has a limit as $n \to \infty$. Again, due to (3.26), for all $t \in \mathbb{R}$ and $x \in S^\Lambda_{\text{fin}}$ there (a.s.) exists $\varepsilon > 0$ such that $I_{t-\varepsilon,t}(x)$ is either the empty set or equal to $\{t\}$. But in the both cases there exists $n_0 \in \mathbb{N}$ such that $\mathbf{X}_{s_n,u}(x) = \mathbf{X}_{s_{n_0},u}(x)$ for all $n \geq n_0$. It follows that $\boldsymbol{\psi}_{\text{mon}}(\mathbf{X}_{s_n,u}(x), Y)$ has a limit as $n \to \infty$ and hence $t \mapsto \mathbf{Y}_{u,t}(Y)$ has (a.s.) left limits. □

Proof of Theorem 5. By Lemma 24 we are left to show that having (1.3), (1.9) and (1.10) implies both (3.19) and that (1.11) almost surely holds. The second assertion is Lemma 1 so we are left to prove the first one.

Let $x \in S^{\Lambda}_{\text{fin}}$ and $m \in \mathcal{G}$. Assume that $m(x) \neq x$. Then, either m changes a value on $\text{supp}(x)$, i.e. there exists $j \in \Lambda$ such that $j \in \mathcal{D}(m) \cap \text{supp}(x)$, or a value outside of $\text{supp}(x)$, i.e. there exists $j \in \Lambda$ such that $j \in \mathcal{D}(m) \cap \text{supp}(x)^{\text{c}}$. But in the latter case, due to the assumption that $m(\underline{0}) = \underline{0}$, there has to exist $i \in \mathcal{R}(m[j]) \cap \text{supp}(x)$ (this was the first observation in the proof of Lemma 1). In particular, $(i, j) \in \mathcal{R}_2(m)$. Hence,

$$\sum_{m \in \mathcal{G}: m(x) \neq x} r_m \leq \sum_{j \in \text{supp}(x)} \sum_{m \in \mathcal{G}} r_m \mathbb{1}_{\mathcal{D}(m)}(j) + \sum_{i \in \text{supp}(x)} \sum_{m \in \mathcal{G}} r_m \sum_{j \in \text{supp}(x)^{\text{c}}} \mathbb{1}_{\mathcal{D}(m)}(j) \mathbb{1}_{\mathcal{R}_2}(i, j) \tag{3.29}$$

and the finiteness of $\text{supp}(x)$ implies together with (1.3) that the first term of the right-hand side above is finite while together with (1.10) it implies that the second term of the right-hand side above is finite. □

The proof above and the proof of Lemma 1 show that the conditions of Theorem 5 are sufficient but indicate that they are not necessary. One particular interesting question is in what sense one can weaken (1.9), i.e. the condition that every local map $m \in \mathcal{G}$ maps $\underline{0}$ to itself. Without the condition (1.9) we lose Lemma 23 and hence have to chose the whole set $\mathcal{H}(S^{\Lambda})$ as the state space of the dual process. Hence, if started in $\mathcal{H}_{-}(S^{\Lambda})$ the dual process can jump into the trap $\{\underline{0}\}$. In order for it to still be a Feller process we have to make sure that the first entrance time of the trap $\{\underline{0}\}$ is almost surely positive. It is straightforward to modify the proofs of Lemma 1 and Theorem 5 to show that having

$$\sup_{i \in \Lambda} \sum_{\substack{m \in \mathcal{G}: \\ m(\underline{0}) = \underline{0}}} r_m \sum_{j \in \Lambda} \mathbb{1}_{\mathcal{D}(m)}(j) \mathbb{1}_{\mathcal{R}_2(m)}(i, j) < \infty \tag{3.30}$$

instead of (1.10) and

$$\sum_{j \in \Lambda} \sum_{\substack{m \in \mathcal{G}: \\ m(\underline{0}) \neq \underline{0}}} r_m \mathbb{1}_{\mathcal{D}(m)}(j) < \infty \tag{3.31}$$

instead of (1.9) suffices to conclude (3.19) and that (1.11) almost surely holds. Note, however, that (3.31) implies that

$$\sum_{\substack{m \in \mathcal{G}: \\ m(\underline{0}) \neq \underline{0}}} r_m < \infty, \tag{3.32}$$

disallowing many natural and interesting cases. Indeed, it seems that the first entrance time of $\{\underline{0}\}$ is a.s. zero in many cases where (3.31) is violated, e.g., for stochastic Ising models or more generally for processes on infinite grids in which zero spins jump with a constant rate to a nonzero value, independent of everything else.

3.2 Dual monotone maps

Before we prove Lemma 7 we state a small fact that will be used both in the proof of Lemma 7 and also at other points in the remainder of this paper. We say that $\mathcal{F}$, a collection of function from one space $\mathcal{X}$ to some other space $\mathcal{Y}$, *separates points* if for $x, x' \in \mathcal{X}$ with $x \neq x'$ there exists a $f \in \mathcal{F}$ such that $f(x) \neq f(x')$.

Lemma 25 (Separation of points) *The collection* $\mathcal{F} := \{\psi_{\text{mon}}(x,\cdot\,) : x \in S^{\Lambda}_{\text{fin}}\}$ *of functions from* $\mathcal{H}_{-}(S^{\Lambda})$ *to* $\mathbb{R}$ *separates points.*

Proof. Let $Y_1, Y_2 \in \mathcal{H}_{-}(S^{\Lambda})$ with $Y_1 \neq Y_2$. Then, by Proposition 2, there exists an $x \in Y_1^{\uparrow} \,\triangle\, Y_2^{\uparrow}$. W.l.o.g. we assume $x \in Y_1^{\uparrow} \backslash Y_2^{\uparrow}$ and we choose $y \in Y_1$ with $y \leq x$. Then $y \notin Y_2^{\uparrow}$ as else $x \in Y_2^{\uparrow}$, and $y \in S^{\Lambda}_{\text{fin}}$ as $Y_1 \subset S^{\Lambda}_{\text{fin}}$. Hence

$$\psi_{\text{mon}}(y, Y_1) = \mathbb{1}_{Y_1^{\uparrow}}(y) \neq \mathbb{1}_{Y_2^{\uparrow}}(y) = \psi_{\text{mon}}(y, Y_2), \tag{3.33}$$

yielding the claim. □

We continue with the proof of Lemma 7.

Proof of Lemma 7. One has that

$$\psi_{\text{mon}}(m(x), Y) = \mathbb{1}_{Y^{\uparrow}}(m(x)) = \mathbb{1}_{m^{-1}(Y^{\uparrow})}(x) \qquad (Y \in \mathcal{H}(S^{\Lambda})). \tag{3.34}$$

With the same argument as in the proof of Lemma 4 the set $m^{-1}(Y^{\uparrow})$ is increasing and open. Hence, by Proposition 2, $m^{-1}(Y^{\uparrow})^{\circ} \in \mathcal{H}(S^{\Lambda})$ and

$$m^{-1}(Y^{\uparrow}) = \left[m^{-1}(Y^{\uparrow})^{\circ}\right]^{\uparrow} \qquad (Y \in \mathcal{H}(S^{\Lambda})). \tag{3.35}$$

Thus, the map $\hat{m}$, defined in (1.29), is dual to m.

To prove the uniqueness assume that also $\tilde{m} : \mathcal{H}(S^{\Lambda}) \to \mathcal{H}(S^{\Lambda})$ is dual to $m : S^{\Lambda} \to S^{\Lambda}$ with respect to ψ_{mon}. Then

$$\psi_{\text{mon}}(x, \tilde{m}(Y)) = \psi_{\text{mon}}(m(x), Y) = \psi_{\text{mon}}(x, \hat{m}(Y)) \qquad (x \in S^{\Lambda},\ Y \in \mathcal{H}(S^{\Lambda})) \tag{3.36}$$

and Lemma 25 implies that $\tilde{m}(Y) = \hat{m}(Y)$ for all $Y \in \mathcal{H}(S^{\Lambda})$, i.e. $\tilde{m} = \hat{m}$.

The last assertion of the lemma follows analogously to the argument in (3.18). □

Next we prove Proposition 8, the main justification for our study of dual maps.

Proof of Proposition 8. Recall that it was shown in the proof of Lemma 24 that having (1.3) and (3.19) implies that, for fixed $u \in \mathbb{R}$ and $Y \in \mathcal{H}_{-}(S^{\Lambda})$, $t \mapsto \mathbf{Y}_{u,t}(Y)$ is (a.s.) indeed a càdlàg function from $(-\infty, u]$ to $\mathcal{H}_{-}(S^{\Lambda})$. Afterwards, in the proof of Theorem 5, we showed that the assumptions of Theorem 5 imply (3.19).

Fix $u \in \mathbb{R}$. We show that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_{-}(S^{\Lambda})$ (a.s.) solves (1.30). The first part of (1.30) follows directly from the definition of $\mathbf{Y}_{s,s}$ and the fact that $\mathbf{X}_{s,s}(y) = y$ for all $y \in S^{\Lambda}$ (compare (1.5) and (1.6)). To show also the second part we introduce further notation. As already mentioned in Section 1.2, due to (1.3) almost surely

$$\nexists t \in \mathbb{R} : |\{m \in \mathcal{G} : (m,t) \in \omega\}| \geq 2. \tag{3.37}$$

Hence, we can almost surely define random maps $\mathfrak{m}^{\omega}_t : S^{\Lambda} \to S^{\Lambda}$ and $\hat{\mathfrak{m}}^{\omega}_t : \mathcal{H}_{-}(S^{\Lambda}) \to \mathcal{H}_{-}(S^{\Lambda})$ for all $t \in \mathbb{R}$ as

$$\mathfrak{m}^{\omega}_t := \begin{cases} m & \text{if } (m,t) \in \omega, \\ \text{id} & \text{else}, \end{cases} \qquad \text{and} \qquad \hat{\mathfrak{m}}^{\omega}_t := \begin{cases} \hat{m} & \text{if } (m,t) \in \omega, \\ \text{id} & \text{else}, \end{cases} \tag{3.38}$$

where id denotes in both cases the identity and $\hat{m}$ is the dual map of $m \in \mathcal{G}$ from Lemma 7. Using the newly introduced notation it follows from the arguments from the proof of Lemma 24 that, for any $t \leq u$,

$$\psi_{\text{mon}}\big(x, \mathbf{Y}_{u,s}(Y)\big) = \psi_{\text{mon}}\big(\mathbf{X}_{s,u}(x), Y\big) \to \psi_{\text{mon}}\big(\mathbf{X}_{t,u} \circ \mathfrak{m}^{\omega}_t(x), Y\big) \quad \text{as } s \nearrow t. \tag{3.39}$$

But, by (1.24) and (1.28),

$$\boldsymbol{\psi}_{\rm mon}\big(\mathbf{X}_{t,u}\circ\mathfrak{m}_t^\omega(x),Y\big)=\boldsymbol{\psi}_{\rm mon}\big(\mathfrak{m}_t^\omega(x),\mathbf{Y}_{u,t}(Y)\big)=\boldsymbol{\psi}_{\rm mon}\big(x,\hat{\mathfrak{m}}_t^\omega(\mathbf{Y}_{u,t}(Y))\big) \tag{3.40}$$

and we conclude from Proposition 3 and the definition of $\boldsymbol{\psi}_{\rm mon}$ that $\mathbf{Y}_{u,s}(Y)\to\hat{\mathfrak{m}}_t^\omega(\mathbf{Y}_{u,t}(Y))$ in $\mathcal{H}_-(S^\Lambda)$ as $s\uparrow t$. As this is just another way of writing the second part of (1.30) we conclude that $(-\infty,u]\ni t\mapsto\mathbf{Y}_{u,t}(Y)\in\mathcal{H}_-(S^\Lambda)$ (a.s.) solves (1.30).

Finally, we prove the uniqueness of the solutions of (1.30). We will show that if $(Y_t)_{t\leq u}$, for fixed $u\in\mathbb{R}$ and $Y\in\mathcal{H}_-(S^\Lambda)$, solves (1.30), then

$$\boldsymbol{\psi}_{\rm mon}(x,Y_s)=\boldsymbol{\psi}_{\rm mon}(X_u,Y)\qquad(s\leq u,\ x\in S^\Lambda_{\rm fin}), \tag{3.41}$$

where $(X_t)_{t\geq s}$ solves (1.5) started at time s in state x. The uniqueness of the solutions of (1.5) together with Lemma 25 then implies the uniqueness of the solutions of (1.30).

We use a strategy similar to the proof of [Swa22, Theorem 6.20]. We equip $S^\Lambda_{\rm fin}\times\mathcal{H}_-(S^\Lambda)$ with the product topology consisting of the discrete topology on $S^\Lambda_{\rm fin}$ and of the topology from Proposition 3 on $\mathcal{H}_-(S^\Lambda)$. By Proposition 3 $\boldsymbol{\psi}_{\rm mon}$ is then a continuous function from $S^\Lambda_{\rm fin}\times\mathcal{H}_-(S^\Lambda)$ to $\{0,1\}$. Fix $u\in\mathbb{R}$ and $Y\in\mathcal{H}_-(S^\Lambda)$ and assume that $(Y_t)_{t\leq u}$ solves (1.30). Fix moreover $s\leq u$ and $x\in S^\Lambda_{\rm fin}$ and assume that $(X_t)_{t\geq s}$ solves (1.5). Then the fact that $(Y_t)_{t\leq u}$ and $(X_t)_{t\geq s}$ are càdlàg implies that

$$[s,u]\ni t\mapsto\boldsymbol{\psi}_{\rm mon}(X_t,Y_t)\in\{0,1\} \tag{3.42}$$

is càdlàg as well. If $(m,t)\in\omega$ for some $t\in(s,u]$, then the evolution equations and (1.28) imply that

$$\boldsymbol{\psi}_{\rm mon}(X_t,Y_t)=\boldsymbol{\psi}_{\rm mon}(m(X_{t-}),Y_t)=\boldsymbol{\psi}_{\rm mon}(X_{t-},\hat{m}(Y_t))=\boldsymbol{\psi}_{\rm mon}(X_{t-},Y_{t-}). \tag{3.43}$$

If there exists no $m\in\mathcal{G}$ such that $(m,t)\in\omega$, then (3.43) holds trivially. The finiteness of $\{0,1\}$ now implies that the function in (3.42) is constant. Plugging in $t=u$ and $t=s$ this implies (3.41) and the proof is complete. □

The final objective of this subsection is to prove Proposition 6. We first state a lemma with an additional property of the dual map $\hat{m}$ from (1.29). In fact, the result below was already stated as part of [SS18, Lemma 29]. However, in [SS18] monotone dual maps are defined via the dual of a partially ordered space (compare Section 1.5). For the readers' convenience below we present a reformulated proof that is adapted to the notation and definitions of the current paper. For any $A\subset S^\Lambda$, we set $\mathrm{supp}(A):=\bigcup_{x\in A}\mathrm{supp}(x)$ and call $\mathrm{supp}(A)$ the support of A.

Lemma 26 (Support of the dual map) *For each continuous monotone map $m:S^\Lambda\to S^\Lambda$, the map $\hat{m}$ from (1.29) satisfies*

$$\mathrm{supp}(\hat{m}(Y))\subset\bigcup_{i\in\mathrm{supp}(Y)}\mathcal{R}(m[i])\qquad(Y\in\mathcal{H}(S^\Lambda)). \tag{3.44}$$

Proof. Let $Y\in\mathcal{H}(S^\Lambda)$. As

$$\mathrm{supp}(\hat{m}(y))=\mathrm{supp}\left(\left[\bigcup_{y\in y}m^{-1}(\{y\}^\uparrow)\right]^\circ\right)\subset\mathrm{supp}\left(\bigcup_{y\in y}m^{-1}(\{y\}^\uparrow)^\circ\right)=\bigcup_{y\in y}\mathrm{supp}(m^{-1}(\{y\}^\uparrow)^\circ) \tag{3.45}$$

and $\mathrm{supp}(Y)=\bigcup_{y\in Y}\mathrm{supp}(y)$, it suffices to show that

$$\mathrm{supp}\big(m^{-1}(\{y\}^\uparrow)^\circ\big)\subset\bigcup_{i\in\mathrm{supp}(y)}\mathcal{R}(m[i])\qquad(y\in S^\Lambda_{\rm fin}). \tag{3.46}$$

Hence, let $y \in S^{\Lambda}_{\mathrm{fin}}$ and assume that $k \in \mathrm{supp}(m^{-1}(\{y\}^{\uparrow})^{\circ})$. By the definition of the support, there then exists an $x \in m^{-1}(\{y\}^{\uparrow})^{\circ}$ with $x(k) \neq 0$. We define $x_{k\rightsquigarrow 0} \in S^{\Lambda}$ via

$$x_{k\rightsquigarrow 0}(m) := \begin{cases} 0 & \text{if } m = k, \\ x(m) & \text{else,} \end{cases} \qquad (m \in \Lambda). \tag{3.47}$$

Then $y \not\leq m(x_{k\rightsquigarrow 0})$ as otherwise the minimality of $x \in m^{-1}(\{y\}^{\uparrow})$ would be violated. Hence, there exists an $i \in \Lambda$ such that $y(i) \leq m(x)(i)$ but $y(i) \not\leq m(x_{k\rightsquigarrow 0})(i)$. This shows that $m(x)(i) \neq m(x_{k\rightsquigarrow 0})(i)$ and hence $k \in \mathcal{R}(m[i])$, and also that $y(i) \neq 0$ so that $i \in \mathrm{supp}(y)$. This establishes (3.46) and hence also (3.44). $\square$

Proof of Proposition 6. By Lemma 21 $\mathbf{X}_{s,u}$ is almost surely a continuous map for all $s \leq u$. Hence, by Lemma 7, the maps $\mathbf{X}_{s,u}$ possess (a.s.) dual maps that we denote by $\hat{\mathbf{X}}_{s,u}$. Then Lemma 26 (using also Lemma 19) implies that the maps $\hat{\mathbf{X}}_{s,u}$ (a.s.) map $\mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$ into itself. Finally, (1.24) and the uniqueness of the dual map in Lemma 7 imply that $\mathbf{Y}_{u,s} = \hat{\mathbf{X}}_{s,u}$.

Fix $u \in \mathbb{R}$, let Y_0 be a random variable with values in $\mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$ that is independent of ω, and let $(Y_t)_{t\geq 0}$ be defined by (1.22). The fact that $(Y_t)_{t\geq 0}$ is a Markov process follows from the fact that it is constructed from a stochastic flow with independent increments (compare the proof of Lemma 24). It remains to show that $(Y_t)_{t\geq 0}$ has (a.s.) càglàd sample paths. As $\mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$ is equipped with the discrete topology, this amounts to showing that $(-\infty, u] \ni t \mapsto \mathbf{Y}_{u,t}(Y) \in \mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$ is (a.s.) piecewise constant and right-continuous for all $Y \in \mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$.

Generalizing the notation $m[i]$ introduced in Subsection 3.1, for any finite set $A \subset \Lambda$ and map $m : S^{\Lambda} \to S^{\Lambda}$, let $m[A]$ denote the map from S^{Λ} to S^{A} defined by $m(x)[A](i) := m(x)(i)$ $(i \in A)$. Recall (1.18). We observe that $\mathbf{X}_{s,u}(x) \in Y^{\uparrow}$ if and only if

$$\exists y \in Y \text{ s.t. } y(i) \leq \mathbf{X}_{s,u}(x)(i) \ \forall i \in \mathrm{supp}(Y). \tag{3.48}$$

It follows from Lemma 20 that for fixed $u \in \mathbb{R}$ and $i \in \Lambda$, the function $s \mapsto \omega_{s,u}(i)$ is piecewise constant and right-continuous and hence the same is true for $s \mapsto \mathbf{X}_{s,u}[i]$. As a consequence, for any finite $A \subset \Lambda$, the map $s \mapsto \mathbf{X}_{s,u}[i]$ is piecewise constant and right-continuous. Applying this to $A = \mathrm{supp}(Y)$, we see from (1.18) and (3.48) that $t \mapsto \mathbf{Y}_{u,t}(Y)$ is piecewise constant and right-continuous for all $Y \in \mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$. $\square$

Before concluding this subsection we add a final small remark: Comparing with [SS18] the reader might notice that the random maps $\{\mathbf{Y}_{u,s}\}_{u\geq s}$ there are not defined as in (1.18), but as limits of concatenations of finitely many dual maps of the maps appearing in the Poison point set ω (compare [SS18, Equation (143) & Proposition 28]). However, based on the proof above, it is not hard to see that we can prove a version of Proposition 8 with $\mathcal{H}_{-}(S^{\Lambda})$ replaced by $\mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$. The uniqueness in the modified version then implies that (a.s.) both approaches yield the same random maps.

4 Upper invariant laws and survival

In this section we prove Proposition 11 and Theorem 12. To do so we prove that $\overline{\nu}$ and $\overline{\mu}$ can be characterized by how they integrate the duality function $\boldsymbol{\psi}_{\mathrm{mon}}$ in the following sense.

Proposition 27 (Characterizing $\overline{\nu}$) *Assume that every map $m \in \mathcal{G}$ is monotone, the rates satisfy (1.3), and that S has a greatest element $\top$. Then the upper invariant law $\overline{\nu}$ of the forward process is uniquely characterized by the relation*

$$\int \boldsymbol{\psi}_{\mathrm{mon}}(x, Y)\, \mathrm{d}\overline{\nu}(x) = \mathbb{P}\big[\mathbf{Y}_{t,0}(Y) \neq \emptyset \ \forall t \geq 0\big] \qquad (Y \in \mathcal{H}_{\mathrm{fin}}(S^{\Lambda})). \tag{4.1}$$

Proposition 28 (Characterizing $\overline{\mu}$) *Assume that every map $m \in \mathcal{G}$ is monotone and satisfies (1.9), and that the rates satisfy (1.3) and (1.10). Then the upper invariant law $\overline{\mu}$ of the dual process is uniquely characterized by the relation*

$$\int \prod_{k=1}^{n} \psi_{\mathrm{mon}}(x_k, Y)\,\mathrm{d}\overline{\mu}(Y) = \mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall t \geq 0,\ k = 1, \ldots, n\big] \quad (n \in \mathbb{N},\ x_1, \ldots, x_n \in S^\Lambda_{\mathrm{fin}}). \tag{4.2}$$

The cornerstone to prove the two propositions above is next lemma. In order to state it we introduce some notation. Throughout the rest of this subsection we denote, for a general topological space $\mathcal{X}$, by $\mathcal{C}(\mathcal{X})$ the space of bounded continuous real functions defined on $\mathcal{X}$. We say that a collection of functions $\mathcal{F} \subset \mathcal{C}(\mathcal{X})$ is *distribution determining* if for probability measures μ, ν on $\mathcal{X}$,

$$\int f(x)\,\mathrm{d}\mu(x) = \int f(x)\,\mathrm{d}\nu(x)\ \forall f \in \mathcal{F} \quad \text{implies} \quad \mu = \nu. \tag{4.3}$$

We cite the following result [Swa22, Lemma 4.37].

Lemma 29 (Application of Stone-Weierstrass) *Let E be a compact metrizable space. If $\mathcal{F} \subset \mathcal{C}(E)$ separates points and is closed under products, then $\mathcal{F}$ is distribution determining.*

For the proof of Proposition 27 we moreover need the following lemma.

Lemma 30 (Clopen increasing subsets) *Let $Y \in \mathcal{H}(S^\Lambda)$. Then Y is finite if and only if $Y^\uparrow$ is closed.*

Proof. Fist assume that $Y \in \mathcal{H}(S^\Lambda)$ is finite, i.e. that there exist $n \in \mathbb{N}_0$ and $y_1, \ldots, y_n \in S^\Lambda_{\mathrm{fin}}$ such that $Y = \{y_1, \ldots, y_n\}$. Then

$$Y^\uparrow = \{y_1\}^\uparrow \cup \cdots \cup \{y_n\}^\uparrow \tag{4.4}$$

is closed as it is the union of finitely many closed sets (compare Lemma 17).

Conversely, assume that $Y^\uparrow$ is closed, and hence clopen (i.e. also open) as $Y^\uparrow \in \mathcal{I}(S^\Lambda)$ by Proposition 2. This implies that $\mathbb{1}_{Y^\uparrow} : S^\Lambda \to \{0, 1\}$ is a continuous function. By Lemma 19 this implies that $\mathcal{R}(\mathbb{1}_{Y^\uparrow})$ from (3.1) is finite. We claim that

$$\mathrm{supp}(Y) \subset \mathcal{R}(\mathbb{1}_{Y^\uparrow}) \tag{4.5}$$

implying the finiteness of Y. To see (4.5) let $i \in \mathrm{supp}(Y)$. Then there exists $y \in Y$ with $i \in \mathrm{supp}(y)$. By the minimality of Y then $y_{i\rightsquigarrow 0} \notin Y^\uparrow$, where $y_{i\rightsquigarrow 0}$, defined in (3.47), denotes the configuration obtained from y by changing the i-th coordinate to 0. Hence $\mathbb{1}_{Y^\uparrow}(y) \neq \mathbb{1}_{Y^\uparrow}(y_{i\rightsquigarrow 0})$ and $i \in \mathcal{R}(\mathbb{1}_{Y^\uparrow})$ implying (4.5). This completes the proof. □

Now we are ready to prove Proposition 27 and Proposition 28.

Proof of Proposition 27. We start by showing that

$$\mathcal{F} := \big\{\psi_{\mathrm{mon}}(\,\cdot\,, Y) : Y \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda) \setminus \{\{\underline{0}\}\}\big\} \subset \mathcal{C}(S^\Lambda) \tag{4.6}$$

is closed under products. Note that $\mathcal{F} \subset \mathcal{C}(S^\Lambda)$ follows from Lemma 30. Let $Y_1, Y_2 \in \mathcal{H}_{\mathrm{fin}}(S^\Lambda)$. Noting that $Y_1^\uparrow \cap Y_2^\uparrow \in \mathcal{I}(S^\Lambda)$ and using Proposition 2 one has that

$$\psi_{\mathrm{mon}}(\,\cdot\,, Y_1)\psi_{\mathrm{mon}}(\,\cdot\,, Y_2) = \mathbb{1}_{Y_1^\uparrow}(\,\cdot\,)\mathbb{1}_{Y_2^\uparrow}(\,\cdot\,) = \mathbb{1}_{Y_1^\uparrow \cap Y_2^\uparrow}(\,\cdot\,) = \psi_{\mathrm{mon}}(\,\cdot\,, (Y_1^\uparrow \cap Y_2^\uparrow)^\circ). \tag{4.7}$$

By Lemma 30 $(Y_1^{\uparrow})^{\mathrm{c}}$ and $(Y_1^{\uparrow})^{\mathrm{c}}$ are open and

$$(Y_1^{\uparrow} \cap Y_2^{\uparrow})^{\mathrm{c}} = (Y_1^{\uparrow})^{\mathrm{c}} \cup (Y_1^{\uparrow})^{\mathrm{c}} \tag{4.8}$$

is open as well. Hence, using Lemma 30 in the converse direction, $(Y_1^{\uparrow} \cap Y_2^{\uparrow})^{\circ}$ is finite.[7] Moreover, if $Y_1 \neq \{\underline{0}\} \neq Y_2$, then clearly also $(Y_1^{\uparrow} \cap Y_2^{\uparrow})^{\circ} \neq \{\underline{0}\}$. Hence, $\mathcal{F}$ is closed under products.

Next we show that $\mathcal{F}$ also separates points. Let $x_1, x_2 \in S^{\Lambda}$ and assume that $x_1 \neq x_2$. Then there has to exist an $i \in \Lambda$ such that $x_1(i) \neq x_2(i)$. Then either $x_1(i) \not\leq x_2(i)$ or $x_2(i) \not\leq x_1(i)$, so interchanging the roles of x_1 and x_2 if necessary, we can w.l.o.g. assume that $x_1(i) \not\leq x_2(i)$. Now

$$\boldsymbol{\psi}_{\mathrm{mon}}\big(x_2, \{\delta_i^{x_1(i)}\}\big) \neq \boldsymbol{\psi}_{\mathrm{mon}}\big(x_1, \{\delta_i^{x_1(i)}\}\big), \tag{4.9}$$

showing that $\mathcal{F}$ separates points and hence $\mathcal{F}$ is distribution determining by Lemma 29.

Let $Y \in \mathcal{H}_{\mathrm{fin}}(S^{\Lambda})$. Since $\emptyset$ is a trap for the dual process we have that

$$\mathbb{P}\big[\mathbf{Y}_{t,0}(Y) \neq \emptyset\big] \searrow \mathbb{P}\big[\mathbf{Y}_{s,0}(Y) \neq \emptyset \ \forall s \geq 0\big] \quad \text{as} \quad t \to \infty. \tag{4.10}$$

Using (1.24) we compute for $t \geq 0$ that

$$\mathbb{E}\big[\boldsymbol{\psi}_{\mathrm{mon}}(\mathbf{X}_{0,t}(\underline{\top}), Y)\big] = \mathbb{E}\big[\boldsymbol{\psi}_{\mathrm{mon}}(\underline{\top}, \mathbf{Y}_{t,0}(Y))\big] = \mathbb{P}\big[\mathbf{Y}_{t,0}(Y) \neq \emptyset\big]. \tag{4.11}$$

Together with (4.10) this implies (4.1). The fact that (4.1) uniquely characterizes $\overline{\nu}$ follows from the fact that $\mathcal{F}$ is distribution determining. □

Proof of Proposition 28. The proof idea is the same as for Proposition 27. This time we want to show that

$$\mathcal{F} := \Big\{\prod_{k=1}^{n} \boldsymbol{\psi}_{\mathrm{mon}}(x_k, \cdot\,) : n \in \mathbb{N},\, x_1, \ldots, x_n \in S^{\Lambda}_{\mathrm{fin}}\Big\} \subset \mathcal{C}(\mathcal{H}(S^{\Lambda})) \tag{4.12}$$

is distribution determining. Note that the continuity of the functions in $\mathcal{F}$ follows directly from Proposition 3 and the definition of $\boldsymbol{\psi}_{\mathrm{mon}}$. The closedness of $\mathcal{F}$ under products follows from its definition. The fact that $\mathcal{F}$ separates points follows from Lemma 25. Now Lemma 29 again implies that $\mathcal{F}$ is distribution determining. Let $\varepsilon > 0$. Since, by Corollary 22, the random maps $\{\mathbf{X}_{s,u}\}_{s \leq u}$ (a.s.) map $\underline{0}$ to itself, we have that

$$\mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0} \ \forall k = 1, \ldots, n\big] \searrow \mathbb{P}\big[\mathbf{X}_{0,s}(x_k) \neq \underline{0} \ \forall s \geq 0,\, k = 1, \ldots, n\big] \quad \text{as} \quad t \to \infty. \tag{4.13}$$

By (1.24) for $t \geq 0$ then

$$\begin{aligned}\mathbb{E}\Big[\prod_{k=1}^{n} \boldsymbol{\psi}_{\mathrm{mon}}(x_k, \mathbf{Y}_{t,0}(Y_{\mathrm{sec}}))\Big] &= \mathbb{E}\Big[\prod_{k=1}^{n} \boldsymbol{\psi}_{\mathrm{mon}}(\mathbf{X}_{0,t}(x_k), Y_{\mathrm{sec}})\Big] \\ &= \mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0} \ \forall k = 1, \ldots, n\big].\end{aligned} \tag{4.14}$$

Together with (4.13) this implies (4.2). As in the previous proof, the fact that (4.2) uniquely characterizes $\overline{\mu}$ follows from the fact that $\mathcal{F}$ is distribution determining. □

Using Proposition 27 the proof of Proposition 11 is now straightforward. The same holds for the proof of Theorem 12 using Proposition 28.

[7] If S is a lattice, then one can check that $(Y_1^{\uparrow} \cap Y_2^{\uparrow})^{\circ} = \{y_1 \vee y_2 : y_1 \in Y_1,\ y_2 \in Y_2\}^{\circ}$, providing an alternative proof that $(Y_1^{\uparrow} \cap Y_2^{\uparrow})^{\circ}$ is finite.

Proof of Proposition 11. Let $\delta_{\underline{0}}$ denote the Dirac measure on $\underline{0} \in S^\Lambda$. As

$$\int \boldsymbol{\psi}_{\rm mon}(x,Y)\ \mathrm{d}\delta_{\underline{0}}(x) = 0 \qquad (Y \in \mathcal{H}_{\rm fin}(S^\Lambda) \setminus \{\{\underline{0}\}\}) \tag{4.15}$$

the fact that $\mathcal{F}$ from (4.6) is distribution determining implies that $\overline{\nu} = \delta_{\underline{0}}$ if and only if $\int \boldsymbol{\psi}_{\rm mon}(x,Y)\ \mathrm{d}\overline{\nu}(x) = 0$ for all $Y \in \mathcal{H}_{\rm fin}(S^\Lambda) \setminus \{\{\underline{0}\}\}$. By (4.1), the latter statement is equivalent to survival of the dual process. Using the fact that $\overline{\nu}$ is an extremal invariant measure [Lig85, Theorem III.2.3], it is easy to see (see [Swa22, Lemma 5.10]) that if $\overline{\nu} \neq \delta_{\underline{0}}$, then $\overline{\nu}$ and $\delta_{\underline{0}}$ are mutually singular. Together, these observations imply the statements of Proposition 11. □

Proof of Theorem 12. Let $\delta_\emptyset$ denote the Dirac measure on $\emptyset \in \mathcal{H}_-(S^\Lambda)$. As

$$\int \prod_{k=1}^n \boldsymbol{\psi}_{\rm mon}(x_k,Y)\ \mathrm{d}\delta_\emptyset(Y) = 0 \qquad (x_1,\dots,x_n \in S^\Lambda_{\rm fin}) \tag{4.16}$$

the fact that $\mathcal{F}$ from (4.12) is distribution determining implies that $\overline{\mu} = \delta_\emptyset$ if and only if $\int \prod_{k=1}^n \boldsymbol{\psi}_{\rm mon}(x_k,Y)\ \mathrm{d}\overline{\mu}(Y) = 0$ for all $x_1,\dots,x_n \in S^\Lambda_{\rm fin}$. By (4.2), the latter statement is equivalent to

$$\mathbb{P}\big[\mathbf{X}_{0,t}(x_k) \neq \underline{0}\ \forall t \geq 0,\ k = 1,\dots,n\big] = 0 \quad (n \in \mathbb{N},\ x_1,\dots,x_n \in S^\Lambda_{\rm fin}), \tag{4.17}$$

which in turn is equivalent to

$$\mathbb{P}\big[\mathbf{X}_{0,t}(x) \neq \underline{0}\ \forall t \geq 0\big] = 0 \quad (x \in S^\Lambda_{\rm fin}). \tag{4.18}$$

The rest of the proof is now the same as the proof of Proposition 11, where one can argue as in [Swa22, Lemma 5.8] to see that $\overline{\mu}$ is an extremal invariant law and then as in [Swa22, Lemma 5.10] to see that $\overline{\mu} \neq \delta_\emptyset$ implies that $\overline{\mu}$ and $\delta_\emptyset$ are mutually singular. □

5 The additive special case

In this section we prove Proposition 13, Proposition 14, Proposition 15, and further study $\mathcal{H}_{\rm pi}(S^\Lambda)$ and ϕ from (1.49) for several examples of S. Without further preparation we prove Proposition 13.

Proof of Proposition 13. It follows from [LS23a, Theorem 2.6] and its proof[8] that the additivity of all maps $m \in \mathcal{G}$ together with (1.3) implies that there (a.s.) exists a stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$, consisting of random maps from $\hat{S}^\Lambda_{\rm fin}$ to itself, satisfying (1.47). Hence, we are left to show that having also (1.10) allows us to extend the stochastic flow $(\mathbf{Z}_{u,s})_{u\geq s}$ to the whole of $\hat{S}^\Lambda$. By [SS18, Lemma 31], as $\mathcal{G}$ now consists of additive local maps, each $m \in \mathcal{G}$ has a unique *additive* dual map with respect to $\boldsymbol{\psi}_{\rm add}$ that we denote by $\bar{m}$. It is known (compare [Swa22, Proposition 2.6] and its proof) that $(\mathbf{Z}_{u,s})_{u\geq s}$ can be extended to the whole of S^Λ if a version of (1.3) holds, where we replace all maps $m \in \mathcal{G}$ with their additive dual maps $\bar{m}$, i.e. if

$$\sup_{i\in\Lambda} \sum_{m\in\mathcal{G}} r_m 1_{\mathcal{D}(\bar{m})}(i)\Big(1 + \sum_{j\in\Lambda} 1_{\mathcal{R}_2(\bar{m})}(j,i)\Big) < \infty. \tag{5.1}$$

[8]Note that [LS23a, Theorem 2.6] requires that S, seen as a monoid, is dual to $\hat{S}$ with a duality function that is a "local version" of $\boldsymbol{\psi}_{\rm add}$. These facts are implied by [LS23b, Lemma 13]. For the exact definitions see the two cited papers.

It follows from [LS23a, Lemma 2.4] that for any $m \in \mathcal{G}$ we can write

$$m(x)(j) = \bigvee_{i \in \Lambda : (i,j) \in \mathcal{R}_2(m)} M_i^j(x(i)) \qquad (j \in \Lambda,\ x \in S^\Lambda), \tag{5.2}$$

where $M_i^j : S \to S$ $(i, j \in \Lambda)$ are some additive maps. From [LS23a, Proposition 2.5] it then follows that we have

$$\bar{m}(\hat{y})(i) = \bigvee_{j \in \Lambda : (i,j) \in \mathcal{R}_2(m)} \bar{M}_i^j(\hat{y}(j)) \qquad (i \in \Lambda,\ \hat{y} \in \hat{S}^\Lambda), \tag{5.3}$$

where $\bar{M}_i^j : \hat{S} \to \hat{S}$ $(i, j \in \Lambda)$ are some additive maps. Hence, clearly $(i, j) \in \mathcal{R}_2(m)$ if and only if $(j, i) \in \mathcal{R}_2(\bar{m})$. Moreover, we infer that

$$\mathcal{D}(\bar{m})^{\mathrm{c}} = \{i \in \Lambda : \{j \in \Lambda : (i,j) \in \mathcal{R}_2(m)\} = \{i\}, M_i^i = \mathrm{id}\} \tag{5.4}$$

If follows that $i \in \mathcal{D}(\bar{m})$ implies that either

(i) there exists a $j \neq i$ such that $(i, j) \in \mathcal{R}_2(m)$,

or

(ii) there exists no $j \neq i$ such that $(i, j) \in \mathcal{R}_2(m)$, but $M_i^i \neq \mathrm{id}$.

Let Λ_1 denote the set of $i \in \Lambda$ satisfying (i) and let Λ_2 denote the set of $i \in \Lambda$ satisfying (ii). Note that in case (i) we have to have $j \in \mathcal{D}(m)$ and in case (ii) we have to have $i \in \mathcal{D}(m)$. We compute that, for all $i \in \Lambda$,

$$\begin{aligned}
&\sum_{m \in \mathcal{G}} r_m 1_{\mathcal{D}(\bar{m})}(i)\Big(1 + \sum_{j \in \Lambda} 1_{\mathcal{R}_2(\bar{m})}(j, i)\Big) \\
&\leq \sum_{m \in \mathcal{G}} r_m \big[1_{\Lambda_1}(i) + 1_{\Lambda_2}(i)\big]\Big(1 + \sum_{j \in \Lambda} 1_{\mathcal{R}_2(m)}(i, j)\Big) \\
&\leq \sum_{m \in \mathcal{G}} r_m 1_{\Lambda_1}(i)\Big(2 + \sum_{j \in \Lambda \setminus \{i\}} 1_{\mathcal{R}_2(m)}(i, j)\Big) + \sum_{m \in \mathcal{G}} 2 r_m 1_{\Lambda_2}(i) \\
&\leq 2 \sum_{m \in \mathcal{G}} r_m \sum_{k \in \Lambda} 1_{\mathcal{D}(m)}(k) + \sum_{m \in \mathcal{G}} r_m \sum_{j \in \Lambda \setminus \{i\}} 1_{\mathcal{D}(m)}(j) 1_{\mathcal{R}_2(m)}(i, j) + 2 \sum_{m \in \mathcal{G}} r_m 1_{\mathcal{D}(m)}(i) \\
&\leq \sum_{m \in \mathcal{G}} r_m \sum_{j \in \Lambda} 1_{\mathcal{D}(m)}(j) 1_{\mathcal{R}_2(m)}(i, j) + 4 \sum_{m \in \mathcal{G}} r_m 1_{\mathcal{D}(m)}(i).
\end{aligned} \tag{5.5}$$

Using now (1.10) for the first term and (1.3) for the second term, we conclude (5.1). This completes the proof. □

We continue with the proof of Proposition 14.

Proof of Proposition 14. Let $\mathcal{I}_{\mathrm{pi}}(S^\Lambda) := \{A \subset S^\Lambda : A^{\mathrm{c}} \text{ is a principal ideal}\}$ and note that $\mathcal{I}_{\mathrm{pi}}(S^\Lambda) \subset \mathcal{I}(S^\Lambda)$ as for all $y \in S^\Lambda$ the set $\{y\}^\downarrow$ is closed by Lemma 17 and decreasing by definition. It is obvious that the map $\hat{y} \mapsto (\{y\}^\downarrow)^{\mathrm{c}}$ is a bijection from $\hat{S}^\Lambda$ to $\mathcal{I}_{\mathrm{pi}}(S^\Lambda)$ and it follows from Proposition 2 that also $\phi : \hat{S}^\Lambda \to \mathcal{H}_{\mathrm{pi}}(S^\Lambda)$ is a bijection.

To see that ϕ is monotone let $\hat{y}_1, \hat{y}_2 \in \hat{S}^\Lambda$ with $\hat{y}_1 \leq \hat{y}_2$. Then

$$y_2 \leq y_1 \text{ in } S^\Lambda \quad \Rightarrow \quad \{y_2\}^\downarrow \subset \{y_1\}^\downarrow \quad \Rightarrow \quad (\{y_1\}^\downarrow)^{\mathrm{c}} \subset (\{y_2\}^\downarrow)^{\mathrm{c}}. \tag{5.6}$$

But the last assertion above says by definition that $\phi(\hat{y}_1) \leq \phi(\hat{y}_2)$ in $\mathcal{H}(S^\Lambda)$. Hence ϕ is monotone.

To prove the continuity of ϕ and ϕ^{-1} we argue as follows. Let $y, y' \in S^\Lambda$. Then, for all $x \in \{y\}^\downarrow$ there exists an $x' \in \{y'\}^\downarrow$ satisfying $x(i) = x'(i)$ for all $i \in \Lambda$ with $y(i) = y'(i)$ which implies $d(x, x') \le d(y, y')$. Likewise, for all $x' \in \{y'\}^\downarrow$ there exists an $x \in \{y\}^\downarrow$ such that $d(x, x') \le d(y, y')$. Hence

$$d_{\rm H}(\{y\}^\downarrow, \{y'\}^\downarrow) \le d(y, y') \qquad (y, y' \in S^\Lambda), \tag{5.7}$$

implying the (Lipschitz) continuity of $y \mapsto \{y\}^\downarrow$ and consequently also of ϕ. Here we use that the maps

$$S^\Lambda \ni y \mapsto \hat{y} \in \hat{S}^\Lambda \quad \text{and} \quad \mathcal{H}_-(S^\Lambda) \ni Y \mapsto (Y^\uparrow)^{\rm c} \in \mathcal{K}_+(S^\Lambda) \tag{5.8}$$

are, due to the definitions of the corresponding metrics, isometries. On the other hand, $d(y, y') \ge 1/3^k$ implies that there exists $i \in \gamma^{-1}(\{1, \dots, k\})$ such that $y(i) \ne y'(i)$. Hence $\{y\}^\downarrow_{\gamma^{-1}(\{1,\dots,k\})} \ne \{y'\}^\downarrow_{\gamma^{-1}(\{1,\dots,k\})}$, where, for $A \subset S^\Lambda$ and $\Delta \subset \Lambda$, $A_\Delta := \{a_\Delta : a \in A\}$ with a_Δ being defined at the beginning of Section 1.2. It follows that there exists either an $x \in \{y\}^\downarrow$ with $d(x, \{y'\}^\downarrow) \ge 1/3^k$ or an $x' \in \{y'\}^\downarrow$ with $d(x', \{y\}^\downarrow) \ge 1/3^k$. It follows that $d_{\rm H}(\{y\}^\downarrow, \{y'\}^\downarrow) \ge 1/3^k$. From this one concludes the continuity of ϕ^{-1}.

Finally, (1.50) follows directly from the definitions of $\boldsymbol{\psi}_{\rm add}$ and ϕ as

$$\boldsymbol{\psi}_{\rm add}(x, \hat{y}) = 1_{(\{y\}^\downarrow)^{\rm c}}(x) = 1_{\phi(\hat{y})^\uparrow}(x) = \boldsymbol{\psi}_{\rm mon}(x, \phi(\hat{y})) \qquad (x \in S^\Lambda,\ \hat{y} \in \hat{S}^\Lambda), \tag{5.9}$$

where we used Proposition 2 in the second equality. □

Making use of (1.50) and the two dualities (the monotone and the additive one) the proof of Proposition 15 is now straightforward.

Proof of Proposition 15. Let $Y \in \mathcal{H}_{\rm pi}(S^\Lambda)$ and $u \le s$. Due to (1.3) and (1.10), $\mathbf{Y}_{u,s}$ and $\mathbf{Z}_{u,s}$ are almost surely well-defined. One now computes that

$$\begin{aligned}\boldsymbol{\psi}_{\rm mon}\big(x, \mathbf{Y}_{u,s}(Y)\big) &= \boldsymbol{\psi}_{\rm mon}\big(\mathbf{X}_{s,u}(x), Y\big) = \boldsymbol{\psi}_{\rm add}\big(\mathbf{X}_{s,u}(x), \phi^{-1}(Y)\big) = \boldsymbol{\psi}_{\rm add}\big(x, \mathbf{Z}_{u,s}(\phi^{-1}(Y))\big) \\ &= \boldsymbol{\psi}_{\rm mon}\big(x, \phi\big(\mathbf{Z}_{u,s}(\phi^{-1}(Y))\big)\big)\end{aligned} \tag{5.10}$$

for all $x \in S^\Lambda$. Here we used (1.24) in the first equality, (1.50) in the second and fourth equality, and (1.47) in the third equality. By Lemma 25, (5.10) implies that

$$\mathbf{Y}_{u,s}(Y) = \phi\big(\mathbf{Z}_{u,s}(\phi^{-1}(Y))\big) \tag{5.11}$$

and, using that ϕ is a bijection from $\hat{S}^\Lambda$ to $\mathcal{H}_{\rm pi}(S^\Lambda)$, we conclude that $\mathbf{Y}_{u,s}(Y) \in \mathcal{H}_{\rm pi}(S^\Lambda)$. □

By now we have proved all results from Section 1.5. An unpleasant feature of Proposition 14 is that the definitions of the space $\mathcal{H}_{\rm pi}(S^\Lambda)$ and the bijection ϕ are rather abstract. In the remainder of this section, we show that in the special case that S is a distributive lattice, one can give a much more concrete description of these objects.

Recall that a lattice S is *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (a, b, c \in S). \tag{5.12}$$

For example, partially ordered sets of the form $S = \{0, \dots, N\}^n$ (equipped with the product order) are distributive lattices ($N, n \in \mathbb{N}$). We call an element[9] $a \in S$ *(join-)irreducible* if

$$a = b \vee c \quad \text{implies} \quad b = a \text{ or } c = a \qquad (b, c \in S). \tag{5.13}$$

[9]Often one excludes 0 from the set of (join-)irreducible elements. However, for our purposes it is convenient to include it.

Let S be a finite lattice. We define $S_{\rm ir} := \{a \in S : a \text{ is irreducible}\}$ and $S^\Lambda_{\rm ir} := \{x \in S^\Lambda : x \text{ is irreducible}\}$. It is easy to see that

$$S^\Lambda_{\rm ir} = \big\{\delta^a_i : i \in \Lambda,\ a \in S_{\rm ir}\big\}. \tag{5.14}$$

We define $\mathcal{H}_1(S^\Lambda) := \{Y \subset S^\Lambda_{\rm ir} : Y^\circ = Y\}$. The following result is the promised less abstract characterization of $\mathcal{H}_{\rm pi}(S^\Lambda)$ in case S is a distributive lattice.

Proposition 31 (Ideals on distributive lattices) *Assume that S is a distributive lattice and let $Y \in \mathcal{H}(S^\Lambda)$. Then $(Y^\uparrow)^{\rm c} \subset S^\Lambda$ is a principal ideal if and only if $Y \in \mathcal{H}_1(S^\Lambda)$, i.e. $\mathcal{H}_{\rm pi}(S^\Lambda) = \mathcal{H}_1(S^\Lambda)$.*

For the proof of Proposition 31 we need the following Lemma. Recall that an ideal of a lattice S is a non-empty, decreasing subset $I \subset S$ that is closed under taking the join.

Lemma 32 (Closed ideals) *An ideal is closed in S^Λ if and only if it is principal.*

Proof. Let $I \subset S^\Lambda$ be an ideal. If I is a principal ideal, i.e. $I = \{y\}^\downarrow$ for some $y \in S^\Lambda$, then I is closed by Lemma 17.

Conversely, assume that I is closed. Recall that a *net* in I is an indexed collection of elements $(y_\alpha)_{\alpha\in\Gamma}$ of I whose index set Γ is equipped with a partial order $\leq$ such that for each $\alpha, \beta \in \Gamma$, there exists a $\gamma \in \Gamma$ such that $\alpha, \beta \leq \gamma$. In particular, if we let $y_x := x$ denote the identity map, then $(y_x)_{x\in I}$ is a net in I. Since I is a closed subset of the compact space S^Λ, it is compact, which implies that each net in I has a convergent subnet. Let $(y_x)_{x\in I'}$ be a convergent subnet of the net we have just described and let y be its limit. The definition of a subnet means that for each $x \in I$ there exists a $x' \in I'$ such that all $I' \ni x'' \geq x'$ satisfy $x'' \geq x$. Using this and the fact that the set $\{z \in I : z \geq x\}$ is closed we see that $y \geq x$ for all $x \in I$. It follows that $I = \{y\}^\downarrow$, i.e., I is principal. □

Proof of Proposition 31. Let $Y \notin \mathcal{H}_1(S^\Lambda)$. Then, by (5.14) there exists a $y \in Y$ with either two non-zero coordinates, i.e. there exist $i, j \in \Lambda$ with $i \neq j$ and $y(i) \neq 0 \neq y(j)$ or with $y(l) \in S \setminus S_{\rm ir}$ for some $l \in \Lambda$. In both cases the minimality of Y implies that $(Y^\uparrow)^{\rm c}$ cannot be an ideal. More precisely, in the first case we have $y = y_{i\rightsquigarrow 0} \vee y_{j\rightsquigarrow 0} \in Y^\uparrow$, where, for $k \in \Lambda$, $y_{k\rightsquigarrow 0}$ denotes the configuration obtained from y by changing the k-th coordinate to 0 defined in (3.47), while the minimality of Y implies that $y_{i\rightsquigarrow 0}, y_{j\rightsquigarrow 0} \notin Y^\uparrow$. In the second case we can write $y(l) = b \vee c$ with $b \neq y(l) \neq c$, change the value of y at l once to b and once to c and run a similar argument as in the first case.

Let now $Y \in \mathcal{H}_1(S^\Lambda)$. Then $(Y^\uparrow)^{\rm c}$ is non-empty and decreasing. Hence, if $(Y^\uparrow)^{\rm c}$ were not a lattice we could find $x_1, x_2 \in (Y^\uparrow)^{\rm c}$ such that $x_1 \vee x_2 \in Y^\uparrow$, i.e. there would exist a $y \in Y$ with the property that $y \leq x_1 \vee x_2$. As the distributivity of S implies the distributivity of S^Λ, we could conclude that

$$y = y \wedge (x_1 \vee x_2) = (y \wedge x_1) \vee (y \wedge x_2). \tag{5.15}$$

As y is irreducible it would follow that either $y \wedge x_1 = y$ or $y \wedge x_2 = y$. But the former implies that that $y \leq x_1$ while the later implies $y \leq x_2$ contradicting that $x_1, x_2 \in (Y^\uparrow)^{\rm c}$. □

To close the subsection we compute the bijection ϕ from (1.49) explicitly for the important example $S = \{0, \ldots, N\}$ $(N \in \mathbb{N})$ equipped with the natural order $0 < 1 < \cdots < N$. We set $\hat{S} = \{0, \ldots, N\}$ with $\hat{y} := \underline{N} - y$ (defined pointwise). One has that

$$(\{y\}^\downarrow)^{\rm c} = \big\{x \in S^\Lambda : \exists i \in \Lambda \text{ s.t. } y(i) < x(i)\big\} \qquad (y \in S^\Lambda), \tag{5.16}$$

and hence

$$\phi(\hat{y}) = \big\{\delta_i^{y(i)+1} : i \in \Lambda \text{ s.t. } y(i) \neq N\big\} = \big\{\delta_i^{N+1-\hat{y}(i)} : i \in \text{supp}(\hat{y})\big\} \qquad (\hat{y} \in \hat{S}^\Lambda). \tag{5.17}$$

Compare also [Fox16, Example 1].

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