Tightness criteria for random compact sets of cadlag paths

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January 5, 2025

Abstract

We give tightness criteria for random variables taking values in the space of all compact sets of cadlag real-valued paths, in terms of both the Skorohod J1 and M1 topologies. This extends earlier work motivated by the study of the Brownian web that was concerned only with continuous paths. In the M1 case, we give a natural extension of our tightness criteria which ensures that non-crossing systems of paths have weak limit points that are also non-crossing. This last result is exemplified through a rescaling of heavy tailed Poisson trees and a more general application to weaves.

MSC 2020. Primary: 60B10; Secondary: 60G07.
Keywords. Tightness, Skorohod topology, J1 topology, M1 topology, path space, noncrossing paths.
Acknowledgments. Work sponsored by GAČR grant 22-12790S.

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1 Main results

1.1 Introduction

A central theme in probability theory is the weak convergence in law of stochastic processes, the canonical example of which is the rescaling of random walks to Brownian motion and, more generally, to α-stable processes. The modern perspective is to treat a stochastic process as a single random variable, a random path, and to consider convergence in terms of probability laws on the space of all possible paths of the process. Paths are usually assumed to be *cadlag* i.e. right-continuous with left limits, with time domain $[0, \infty)$ and values in some Polish space M. The space $D_{[0,\infty)}(M)$ of such paths is often equipped with Skorohod's J1 topology, which is commonly known as 'the' Skorohod topology. For real valued processes, the slightly weaker Skorohod M1 topology is sometimes required instead e.g. [Whi02, DGM24]. Proofs of weak convergence typically involve establishing that (1) a sequence of probability laws on the space of all paths is tight and (2) has a unique weak cluster point; the classical works [EK86] and [Bil99] are standard references for this approach. For this reason, tightness criteria are of central importance.

The introduction of the Brownian web [FINR04] started interest in random variables that are not a single random path, but rather a random compact *set of* paths. The framework for doing so established by [FINR04] for the Brownian web (often known as 'the Brownian web topology') treats continuous paths. More recently random compact sets whose elements are paths with jumps have been considered [MRV19, FS24], along with surprising tightness failures for continuous paths that are related to the appearance of a small fraction of jumps in the limit [BMRV06, SSY21]. This naturally asks for tightness criteria for random sets of cadlag paths, which is the focus of the present article.

We consider systems in which different paths are typically defined on different time domains. The Brownian web, for example, covers space-time $\mathbb{R} \times \mathbb{R}$ with continuous paths π for which the time domains are of the form $[s, \infty)$, and $s \in \mathbb{R}$ is called the starting time of π. This necessitates an underlying topology (on individual paths) that allows for the convergence of a sequence of paths that may all have different time domains. In the work of [FINR04], for continuous paths, such convergence can roughly be described as locally uniform convergence of functions plus convergence of the starting times. In [FS23] we generalised this principle by introducing a space of cadlag paths for which the time domains are *arbitrary* closed subsets of the real line. Although our focus is on random sets of paths, we note that this generalization is natural in even the most basic examples e.g. we may then express the rescaling of discrete time random walks to continuous time processes without involving (linear or otherwise) interpolation in between discrete times of the walk.

In [FS23] we showed that the space introduced therein of cadlag paths (with arbitrary closed subsets as time domains) was Polish, for both the J1 and M1 cases. Our present paper builds on this work by deriving tightness criteria for random compact subsets of this space, in the case where paths are real valued. The main result is Theorem 1.2 below. It provides a robust criterion that involves checking a single property, based on how frequently paths cross small intervals of $\mathbb{R}$.

When working with cadlag paths on varying time domains, one complication is that the property of two paths not crossing each other is not automatically preserved by taking limits. This complication arises from jumps that may form at the starting times of limiting paths; we remark that our state space permits this possibility, because it is necessarily a feature of treating random sets of cadlag paths as compact sets. The theory that has developed around the Brownian web has shown that systems of non-crossing paths are substantially easier to work with than systems with crossing paths. With this in mind, in Theorem 1.3 we note a natural analogue of our main result which implies not only tightness, but also that all weak limit points are non-crossing.

We give the proofs in Section 2 followed by two applications of our results in Section 3. Sections 2 and 3 may be read independently of one another. In Section 3.1 we give tightness criteria for weaves – a *weave*, introduced in [FS24], is a random compact subset $\mathcal{A} \subseteq \Pi^{\uparrow}$ that is non-crossing and for which, with probability one, for all $z \in \mathbb{R}^2$ there exists $\pi \in \mathcal{A}$ such that $z \in \overline{\pi}$ i.e. the paths cover space-time. We show that, for weaves, tightness essentially comes down to tightness of the motion of a single particle. In Section 3.2 we exemplify this result on a sequence of rescaled Poisson trees in which the limiting motion of a single path is an α-stable process.

1.2 Path space

We begin by recalling the necessary definitions and related background from [FS23]. Let $\overline{\mathbb{R}} := [-\infty, \infty]$. There exists a unique metrisable topology on

$$\mathbb{R}^2_{\rm c} := (\overline{\mathbb{R}} \times \mathbb{R}) \cup \big\{(*, -\infty), (*, +\infty)\big\} \tag{1.1}$$

such that $\mathbb{R}^2_{\rm c}$ is compact and a sequence $(x_n, t_n) \in \mathbb{R}^2_{\rm c}$ converges to a limit (x, t) if and only if

(i) $t_n \to t$ in the topology on $\overline{\mathbb{R}}$,

(ii) if $t \in \mathbb{R}$, then $x_n \to x$ in the topology on $\overline{\mathbb{R}}$.

The space $R^2_{\rm c}$ has earlier been introduced in [FINR04]. We can think of $\mathbb{R}^2_{\rm c}$ as being obtained from the space $\overline{\mathbb{R}}^2$ after squeezing the sets $\overline{\mathbb{R}} \times \{\pm\infty\}$ into the single points $(*, \pm\infty)$. For this reason, we call $\mathbb{R}^2_{\rm c}$ the *squeezed space*. We let $d_{\rm sqz}$ denote any metric generating the topology on $\mathbb{R}^2_{\rm c}$.

Let I be a closed subset of $\mathbb{R}$, let

$$I^- := \big\{t \in I : (t - \varepsilon, t) \cap I \neq \emptyset \ \forall \varepsilon > 0\big\} \quad \text{and} \quad I^+ := \big\{t \in I : (t, t + \varepsilon) \cap I \neq \emptyset \ \forall \varepsilon > 0\big\} \tag{1.2}$$

denote the sets of points in I that can be approximated from the right or left, respectively, and let $I \ni t \mapsto \pi(t\pm) \in \overline{\mathbb{R}}$ be right- and left-continuous functions such that

$$\pi(t-) = \lim_{s \uparrow t} \pi(s+) \quad (t \in I^-) \quad \text{and} \quad \pi(t+) = \lim_{s \downarrow t} \pi(s-) \quad (t \in I^+). \tag{1.3}$$

We call the triple consisting of I and the functions $t \mapsto \pi(t-)$ and $t \mapsto \pi(t+)$ a *path* and we say that I is the *domain* of the path. We let Π denote the set of all paths. For $x, z \in \overline{\mathbb{R}}$ we write $[x, z] := \{y \in \overline{\mathbb{R}} : x \wedge z \leq y \leq x \vee z\}$, and we set

$$\begin{aligned} \pi &:= \big\{(x, t) : t \in I,\ x \in \{\pi(t-), \pi(t+)\}\big\}, \\ \overline{\pi} &:= \big\{(x, t) : t \in I,\ x \in [\pi(t-), \pi(t+)]\big\}. \end{aligned} \tag{1.4}$$

We call π the *closed graph* and $\overline{\pi}$ the *filled graph* of π. We equip these sets with a total order $\preceq$ such that two elements $(x, s), (y, t) \in \pi, \overline{\pi}$ are strictly ordered $(x, s) \prec (y, t)$ if and only if either $s < t$, or $s = t$ and x lies closer to $\pi(t-)$ than y. Note that the set I as well as the right- and left-continuous functions $t \mapsto \pi(t+)$ and $t \mapsto \pi(t-)$ can be read off from the closed or filled graph together with the total order $\preceq$. For this reason, we often identify a path with its closed graph and denote a path simply as π and its domain by $I_\pi := I$. We set

$$\pi^* := \pi \cup \big\{(*, -\infty), (*, +\infty)\big\} \quad \text{and} \quad \overline{\pi}^* := \overline{\pi} \cup \big\{(*, -\infty), (*, +\infty)\big\}. \tag{1.5}$$

By [FS23, Lemma 3.1] π^* and $\overline{\pi}^*$ are compact subsets of the squeezed space $\mathbb{R}^2_{\rm c}$. We extend the total order $\preceq$ to π^* and $\overline{\pi}^*$ in such a way that $(-\infty,*)$ is the minimal element and $(\infty,*)$ the maximal element.

A *correspondence* between two sets A,B is a set $C\subset A\times B$ such that

$$\forall a\in A\ \exists b\in B \text{ s.t. } (a,b)\in C \quad\text{and}\quad \forall b\in B\ \exists a\in A \text{ s.t. } (a,b)\in C. \tag{1.6}$$

We let $\mathrm{Corr}(A,B)$ denote the set of all correspondences between A and B. If $\mathcal{X}$ is a metrisable topological space, then we let $\mathcal{K}_+(\mathcal{X})$ denote the space of nonempty compact subsets of $\mathcal{X}$. If d is a metric generating the topology on $\mathcal{X}$, then the corresponding *Hausdorff metric* on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\rm H}(K_1,K_2) := \inf_{z_1\in K_1} d(z_1,K_2)\vee \inf_{z_2\in K_2} d(z_2,K_1) = \inf_{C\in\mathrm{Corr}(K_1,K_2)}\ \sup_{(z_1,z_2)\in C} d(z_1,z_2), \tag{1.7}$$

where as usual $d(z,K):=\inf_{z'\in K} d(z,z')$ denotes the distance of a point to a set. It follows from [SSS14, Lemma B.1] that the topology on $\mathcal{K}_+(\mathcal{X})$ only depends on the topology on $\mathcal{X}$ and not on the choice of the metric. We call this the *Hausdorff topology* on $\mathcal{K}_+(\mathcal{X})$. We note that [SSS14, Lemmas B.2 and B3] show that $\mathcal{K}_+(\mathcal{X})$ is Polish if $\mathcal{X}$ is Polish, and compact if $\mathcal{X}$ is compact.

For paths $\pi_1,\pi_2\in\Pi$, we let $\mathrm{Corr}_+(\pi_1^*,\pi_2^*)$ denote the set of correspondences C between π_1^* and π_2^* that are *monotone* in the sense that

$$\text{there are no } (z_1,z_2),(z_1',z_2')\in C \text{ such that } z_1\prec_1 z_1' \text{ and } z_2'\prec_2 z_2. \tag{1.8}$$

If $d_{\rm sqz}$ is a metric generating the topology on the squeezed space $\mathbb{R}^2_{\rm c}$, then we define a corresponding metric $d_{\rm J1}$ on the path space Π by

$$d_{\rm J1}(\pi_1,\pi_2) := \inf_{C\in\mathrm{Corr}_+(\pi_1^*,\pi_2^*)}\ \sup_{(z_1,z_2)\in C} d_{\rm sqz}(z_1,z_2) \qquad \big(\pi_1,\pi_2\in\Pi\big). \tag{1.9}$$

We define $d_{\rm M1}(\pi_1,\pi_2)$ in the same way, but with $\mathrm{Corr}_+(\pi_1^*,\pi_2^*)$ replaced by $\mathrm{Corr}_+(\overline{\pi}_1^*,\overline{\pi}_2^*)$, the set of monotone correspondences between the compactified filled graphs $\overline{\pi}_1^*$ and $\overline{\pi}_2^*$. It follows from [FS23, Thm 2.10] that the topologies generated by these metrics do not depend on the choice of the metric $d_{\rm sqz}$ on $\mathbb{R}^2_{\rm c}$. By [FS23, Prop 3.3], the space Π, equipped with either topology, is Polish.[1] As explained in [FS23, Subsection 3.4], the topology generated by $d_{\rm J1}$ corresponds to Skorohod's J1-topology while $d_{\rm M1}$ generates Skorohod's M1-topology. Or rather, these are generalisations of these two classical topologies that allow for convergence of sequences of paths that need not all have the same domain.

We let $\Pi_{\rm c}$ and $\Pi^{|}$ denote the subspaces of Π consisting of *continuous* and *connected* paths, respectively, defined as

$$\begin{aligned}\Pi_{\rm c} &:= \big\{\pi\in\Pi : \pi(t-)=\pi(t+)\ \forall t\in I_\pi\big\},\\ \Pi^{|} &:= \big\{\pi\in\Pi : I \text{ is an interval}\big\},\end{aligned} \tag{1.10}$$

and we naturally write $\Pi^{|}_{\rm c} := \Pi_{\rm c}\cap\Pi^{|}$. We also write $\Pi^{\uparrow}$ and $\Pi^{\downarrow}$ for the subsets of $\Pi^{|}$ consisting of paths whose domain is an interval that is unbounded from above and below, respectively, and we let $\Pi^{\updownarrow} := \Pi^{\uparrow}\cap\Pi^{\downarrow}$ denote the set of bi-infinite paths. Then $\Pi^{\uparrow}_{\rm c}$ is the classical path space of the Brownian web [FINR04]. For paths $\pi\in\Pi_{\rm c}$ we write $\pi(t):=\pi(t-)=\pi(t+)$ $(t\in I_\pi)$. As a

[1] When we say that Π is Polish we mean that Π is separable and there exists a complete metric d generating the topology on Π. This does not imply that the metrics $d_{\rm J1}$ and $d_{\rm M1}$ are complete, which, indeed, they are not.

result of [FS23, Prop 3.4], the metrics d_{J1} and d_{M1} generate the same topology on Π_{c}. It follows from [FS23, Lemma 3.5] that $\Pi_{\mathrm{c}}^{\downharpoonleft}$ is closed as a subset of Π in the J1-topology. We note that paths in $\Pi^{\uparrow}$ can make a jump at their starting time. This is different from the usual conventions for the space of cadlag functions defined on $[0,\infty)$ but will be crucial for Theorem 1.2 below as well as in future applications we have in mind.

1.3 Tightness for random compact sets of paths

Let Π be the path space introduced in the previous subsection, equipped with either the J1 or M1 topology, and let $\mathcal{K}_+(\Pi)$ be the space of nonempty compact subsets of Π, equipped with the Hausdorff topology. We note that since Π is Polish under both the J1 and M1 topologies, by [FS23, Lemma 2.7] the resulting topology on $\mathcal{K}_+(\Pi)$ is also Polish in either case. We will be interested in weak convergence of probability measures on $\mathcal{K}_+(\Pi)$ with respect to both the J1 and M1 topologies on Π. General topology tells us[2] that a sequence of probability measures μ_n on $\mathcal{K}_+(\Pi)$ converges weakly to a limit if and only if the set $\{\mu_n : n\in\mathbb{N}\}$ is precompact and μ is its only cluster point. Since $\mathcal{K}_+(\Pi)$ is Polish, Prohorov's theorem tells us that a family $\{\mu_\gamma : \gamma\in\Gamma\}$ of probability measures on $\mathcal{K}_+(\Pi)$ is precompact if and only if it is *tight*, i.e., for all $\varepsilon>0$, there exists a compact $\mathcal{C}\subset\Pi$ such that

$$\sup_{\gamma\in\Gamma}\mu_\gamma\big(\Pi\backslash\mathcal{C}\big)\leq\varepsilon. \tag{1.11}$$

We are therefore naturally interested in tightness criteria for families of probability measures on $\mathcal{K}_+(\Pi)$, with respect to the J1 and M1 topologies on Π. As a warm-up, we discuss tightness on $\mathcal{K}_+(\Pi_{\mathrm{c}})$, where Π_{c}, defined in (1.10), is the space of continuous paths. As already mentioned, the metrics d_{J1} and d_{M1} generate the same topology on Π_{c}.

For each $\pi\in\Pi$ and real $T,\delta>0$, we set

$$\begin{aligned}\Delta^2_{T,\delta}(\pi):=\big\{(x,y):\exists-T\leq s\leq t\leq T\text{ s.t. }t-s\leq\delta\\ \text{and }(x,s),(y,t)\in\pi,\ (x,s)\preceq(y,t)\big\}.\end{aligned} \tag{1.12}$$

For each $T,\delta,\varepsilon>0$ and $r\in\mathbb{R}$, we define sets of paths by

$$\begin{aligned}\mathcal{S}^+_{T,\delta,\varepsilon,r}&:=\big\{\pi\in\Pi:\exists(x,y)\in\Delta^2_{T,\delta}(\pi)\text{ s.t. }x\leq r,\ r+\varepsilon\leq y\big\},\\ \mathcal{S}^-_{T,\delta,\varepsilon,r}&:=\big\{\pi\in\Pi:\exists(x,y)\in\Delta^2_{T,\delta}(\pi)\text{ s.t. }y\leq r,\ r+\varepsilon\leq x\big\},\end{aligned} \tag{1.13}$$

and we set $\mathcal{S}^2_{T,\delta,\varepsilon,r}:=\mathcal{S}^+_{T,\delta,\varepsilon,r}\cup\mathcal{S}^-_{T,\delta,\varepsilon,r}$. The set $\mathcal{S}^+_{T,\delta,\varepsilon,r}$ comprises paths that cross $[r,r+\varepsilon]$ from left to right within time $[-T,T]$, with the condition that once the crossing has begun it takes time less than δ to complete. The set $\mathcal{S}^-_{T,\delta,\varepsilon,r}$ corresponds to crossings from right to left. Our first result is an extension of earlier work [FINR04, Prop B1] which therein is restricted to the space $\Pi^{\uparrow}_{\mathrm{c}}$.

Theorem 1.1 (Tightness criterion for sets of continuous paths) *Let $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ be a family of random variables with values in $\mathcal{K}_+(\Pi_{\mathrm{c}})$. Then the laws $(\mu_\gamma)_{\gamma\in\Gamma}$ with $\mu_\gamma:=\mathbb{P}[\mathcal{A}_\gamma\in\cdot\,]$ are tight with respect to the topology on Π_{c} if and only if*

$$\lim_{\delta\to0}\sup_{\gamma\in\Gamma}\mathbb{P}\big[\mathcal{S}^2_{T,\delta,\varepsilon,r}\cap\mathcal{A}_\gamma\neq\emptyset\big]=0\quad\forall T,\varepsilon>0,\ r\in\mathbb{R}. \tag{1.14}$$

[2]In any metrisable space, a sequence x_n converges to a limit x if and only if the set $\{x_n : n\in\mathbb{N}\}$ is precompact (i.e., its closure is compact) and x is its only cluster point (i.e., subsequential limit point).

We now turn our attention to paths with jumps. For each $\pi \in \Pi$ and real $T, \delta > 0$, we set

$$\begin{aligned}\Delta^3_{T,\delta}(\pi) := \big\{(x,y,z) : \exists -T \le s \le t \le u \le T \text{ s.t. } u - s \le \delta \\ \text{and } (x,s),(y,t),(z,u) \in \pi,\ (x,s) \preceq (y,t) \preceq (z,u)\big\}.\end{aligned} \tag{1.15}$$

For each $T, \delta, \varepsilon > 0$ and $r \in \mathbb{R}$, we define sets of paths by

$$\begin{aligned}
\mathcal{S}^{+-}_{T,\delta,\varepsilon,r} &:= \big\{\pi \in \Pi : \exists (x,y,z) \in \Delta^3_{T,\delta}(\pi) \text{ s.t. } x, z \le r,\ r + \varepsilon \le y\big\},\\
\mathcal{S}^{-+}_{T,\delta,\varepsilon,r} &:= \big\{\pi \in \Pi : \exists (x,y,z) \in \Delta^3_{T,\delta}(\pi) \text{ s.t. } y \le r,\ r + \varepsilon \le x, z\big\},\\
\mathcal{S}^{++}_{T,\delta,\varepsilon,r} &:= \big\{\pi \in \Pi : \exists (x,y,z) \in \Delta^3_{T,\delta}(\pi) \text{ s.t. } x \le r,\ r + \varepsilon \le y \le r + 2\varepsilon,\ r + 3\varepsilon \le z\big\},\\
\mathcal{S}^{--}_{T,\delta,\varepsilon,r} &:= \big\{\pi \in \Pi : \exists (x,y,z) \in \Delta^3_{T,\delta}(\pi) \text{ s.t. } z \le r,\ r + \varepsilon \le y \le r + 2\varepsilon,\ r + 3\varepsilon \le x\big\},
\end{aligned} \tag{1.16}$$

and we set

$$\mathcal{S}^{\mathrm{J}}_{T,\delta,\varepsilon,r} := \mathcal{S}^{++}_{T,\delta,\varepsilon,r} \cup \mathcal{S}^{--}_{T,\delta,\varepsilon,r} \quad \text{and} \quad \mathcal{S}^{\mathrm{M}}_{T,\delta,\varepsilon,r} := \mathcal{S}^{+-}_{T,\delta,\varepsilon,r} \cup \mathcal{S}^{-+}_{T,\delta,\varepsilon,r}. \tag{1.17}$$

The set $\mathcal{S}^{+-}_{T,\delta,\varepsilon,r}$ comprises paths that cross $[r, r+\varepsilon]$ from left to right, and then back again from right to left, during time $[-T, T]$ with the whole exercise taking time less that δ; whilst $\mathcal{S}^{-+}_{T,\delta,\varepsilon,r}$ corresponds to similar movement from right to left. The set $\mathcal{S}^{++}_{T,\delta,\varepsilon,r}$ comprises paths that cross $[r, r+\varepsilon]$ from left to right, then take a value in $[r+\varepsilon, r+2\varepsilon]$, then cross $[r+2\varepsilon, r+3\varepsilon]$ from left to right, during time $[-T, T]$ with the whole exercise taking time less that δ; whilst $\mathcal{S}^{--}_{T,\delta,\varepsilon,r}$ corresponds to similar movement in opposite directions.

Our main result is the following theorem. Roughly, this says that the laws of $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ are tight with respect to the J1 topology on Π if the sets $\mathcal{A}_\gamma$ do not contain paths that make two jumps in an arbitrarily brief time after each other. For the M1 topology, it suffices to look only at two jumps in opposite directions.

Theorem 1.2 (Tightness criteria for sets of cadlag paths) *Let $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ be a family of random variables with values in $\mathcal{K}_+(\Pi)$. Then the laws $(\mu_\gamma)_{\gamma\in\Gamma}$ with $\mu_\gamma := \mathbb{P}[\mathcal{A}_\gamma \in \cdot\,]$ are tight with respect to the J1 topology on Π if and only if*

$$\begin{aligned}
&\text{(i)}\ \lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{S}^{\mathrm{M}}_{T,\delta,\varepsilon,r} \cap \mathcal{A}_\gamma \neq \emptyset\big] = 0 \quad \forall T, \varepsilon > 0,\ r \in \mathbb{R},\\
&\text{(ii)}\ \lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{S}^{\mathrm{J}}_{T,\delta,\varepsilon,r} \cap \mathcal{A}_\gamma \neq \emptyset\big] = 0 \quad \forall T, \varepsilon > 0,\ r \in \mathbb{R}.
\end{aligned} \tag{1.18}$$

The laws $(\mu_\gamma)_{\gamma\in\Gamma}$ with $\mu_\gamma := \mathbb{P}[\mathcal{A}_\gamma \in \cdot\,]$ are tight with respect to the M1 topology on Π if and only if (1.18) (i) holds.

Note that if the laws of the $\mathcal{A}_\gamma$'s are invariant under translations then it suffices to check (1.18) (i) and (ii) for $r = 0$. If the laws are invariant under reflection then we can replace $\mathcal{S}^{\mathrm{M}}_{T,\delta,\varepsilon,r}$ by $\mathcal{S}^{+-}_{T,\delta,\varepsilon,r}$ and $\mathcal{S}^{\mathrm{J}}_{T,\delta,\varepsilon,r}$ by $\mathcal{S}^{++}_{T,\delta,\varepsilon,r}$.

1.4 Noncrossing sets of paths

Recall from (1.10) that $\Pi^{|}$ is the space of "connected" paths, whose domain is an interval, and $\Pi^{\updownarrow}$ is the space of bi-infinite paths. Note that $\pi \in \Pi^{|}$ if and only if its filled graph $\overline{\pi}$ as defined in (1.4) is connected in the topological sense. We say that a path π' *extends* a path π if $\overline{\pi} \subset \overline{\pi}'$. Following [FS23], for $\pi_1, \pi_2 \in \Pi^{|}$, we write $\pi_1 \lhd \pi_2$ if π_1 and π_2 can be extended to bi-infinite paths $\pi'_1, \pi'_2 \in \Pi^{\updownarrow}$ such that $\pi'_1(t\pm) \le \pi'_2(t\pm)$ for all $t \in \mathbb{R}$. We note that in spite of the suggestive notation, this relation is not transitive, i.e., $\pi_1 \lhd \pi_2 \lhd \pi_3$ does not imply $\pi_1 \lhd \pi_3$. We say that a set $\mathcal{A} \subset \Pi^{|}$ is *noncrossing* if each $\pi_1, \pi_2 \in \mathcal{A}$ satisfy $\pi_1 \lhd \pi_2$ or $\pi_2 \lhd \pi_1$ (or both). Throughout the present subsection, we equip $\Pi^{|}$ with the M1 topology and we equip

$$\mathcal{K}_{\rm nc}(\Pi^{|}) := \big\{\mathcal{A} \in \mathcal{K}_+(\Pi^{|}) : \mathcal{A} \text{ is noncrossing}\big\} \tag{1.19}$$

with the corresponding Hausdorff topology. We will prove a tightness criterion for random variables with values in $\mathcal{K}_{\rm nc}(\Pi^{|})$. We note that when we say that a collection of probability measures is *tight on* $\mathcal{K}_{\rm nc}(\Pi^{|})$, we mean that the compact subsets that occur in the definition of tightness (see (1.11)) are subsets of $\mathcal{K}_{\rm nc}(\Pi^{|})$ (and not of some larger space). In particular, this implies that each weak cluster point is concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$, which would not follow from the tightness criterion with respect to the M1 topology of Theorem 1.2.

Nevertheless, the tightness criterion for noncrossing sets of paths turns out to be very similar to condition (1.18) (i) (though a bit stronger). To formulate it, for each $\pi_1, \pi_2 \in \Pi$ and real $T, \delta > 0$, we set

$$\begin{aligned}\Delta^2_{T,\delta}(\pi_1,\pi_2) := \big\{(x_1,y_1,x_2,y_2) : \exists -T \leq s_i \leq t_i \leq T \text{ s.t. } (x_i,s_i),(y_i,t_i) \in \pi_i,\\ (x_i,s_i) \preceq (y_i,t_i)\ (i=1,2) \text{ and } (t_1 \vee t_2) - (s_1 \wedge s_2) \leq \delta\big\},\end{aligned} \tag{1.20}$$

and we define

$$\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} := \big\{(\pi_1,\pi_2) \in \Pi^2 : \exists (x_1,y_1,x_2,y_2) \in \Delta^2_{T,\delta}(\pi_1,\pi_2) \text{ s.t. } x_1,y_2 \leq r,\ r+\varepsilon \leq y_1,x_2\big\}. \tag{1.21}$$

Theorem 1.3 (Tightness criterion for noncrossing sets of paths) *Equip* $\Pi^{|}$ *with the M1 topology and* $\mathcal{K}_{\rm nc}(\Pi^{|})$ *with the corresponding Hausdorff topology. Let* $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ *be a family of random variables with values in* $\mathcal{K}_{\rm nc}(\Pi^{|})$*. Then the laws* $(\mu_\gamma)_{\gamma\in\Gamma}$ *with* $\mu_\gamma := \mathbb{P}[\mathcal{A}_\gamma \in \cdot\,]$ *are tight on* $\mathcal{K}_{\rm nc}(\Pi^{|})$ *if and only if*

$$\lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_\gamma \times \mathcal{A}_\gamma) \neq \emptyset\big] = 0 \quad \forall T,\varepsilon > 0,\ r \in \mathbb{R}. \tag{1.22}$$

In words, the event $\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_\gamma \times \mathcal{A}_\gamma) \neq 0$ occurs when $\mathcal{A}_\gamma$ contains a path that crosses $[r, r+\varepsilon]$ in one direction, and another (perhaps different) path that crosses $[r, r+\varepsilon]$ in the opposite direction, with both events taking place within a single time window of length δ between times $-T$ and T. As already discussed at the end of Section 1.1, in Section 3 we give some applications of Theorem 1.3.

It is easy to see that (1.22) implies (1.18) (i) (see Lemma 2.18 below), but the converse implication does not hold in general. For sets of bi-infinite paths, the two conditions are equivalent. This is a consequence of Theorems 1.2 and 1.3 and the following lemma.

Lemma 1.4 (Sets of bi-infinite paths) *Let* Π *be equipped with the M1 topology and* $\mathcal{K}_+(\Pi)$ *with the corresponding Hausdorff topology. Then* $\mathcal{K}_{\rm nc}(\Pi^{\updownarrow})$ *is a closed subset of* $\mathcal{K}_+(\Pi)$*.*

2 Proofs

2.1 Convergence criteria

Let $\mathcal{X}$ be a metrisable space, let d be any metric generating the topology, and let $\mathcal{K}_+(\mathcal{X})$ be the space of nonempty compact subsets of $\mathcal{X}$, equipped with the Hausdorff topology. We recall that by [SSS14, Lemma B.3], if $\mathcal{X}$ is compact, then so is $\mathcal{K}_+(\mathcal{X})$. We cite the following lemma from [SSS14, Lemma B.1].

Lemma 2.1 (Convergence in the Hausdorff topology) *Let $K_n, K \in \mathcal{K}_+(\mathcal{X})$. Then $K_n \to K$ in the Hausdorff topology if and only if there exists a $C \in \mathcal{K}_+(\mathcal{X})$ such that $K_n \subset C$ for all n and*

$$\begin{aligned} K &\subset \big\{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x_n \to x\big\} \\ &\subset \big\{x \in \mathcal{X} : \exists x_n \in K_n \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\big\} \subset K. \end{aligned} \tag{2.1}$$

Assume that $\mathcal{X}$ is separable and locally compact. Let $\mathcal{X}_\infty := \mathcal{X} \cup \{\infty\}$ denote its one-point compactification. It is well-known that $\mathcal{X}_\infty$ is metrisable. We observe that $\mathcal{C} := \{A \in \mathcal{K}_+(\mathcal{X}_\infty) : \infty \in A\}$ is a closed subset of $\mathcal{K}_+(\mathcal{X}_\infty)$ and hence compact. The map $A \mapsto A_\infty := A \cup \{\infty\}$ is a bijection from the space $\mathrm{Cl}(\mathcal{X})$ of closed subsets of $\mathcal{X}$ to $\mathcal{C}$. We use this to equip $\mathrm{Cl}(\mathcal{X})$ with the metric d_c defined as

$$d_\mathrm{c}(A, B) := d_\mathrm{H}\big(A_\infty, B_\infty\big) \qquad \big(A, B \in \mathrm{Cl}(\mathcal{X})\big), \tag{2.2}$$

where d_H is the Hausdorff metric associated with the metric d on $\mathcal{X}$. By our previous observation, $\big(\mathrm{Cl}(\mathcal{X}), d_\mathrm{c}\big)$ is compact. We call the topology generated by d_c the *local Hausdorff topology*. The following lemma shows that this topology does not depend on the choice of the metric d on $\mathcal{X}$.

Lemma 2.2 (The local Hausdorff topology) *Let $A_n, A \in \mathrm{Cl}(\mathcal{X})$. Then $A_n \to A$ in the local Hausdorff topology if and only if*

$$\begin{aligned} A &\subset \big\{x \in \mathcal{X} : \exists x_n \in A_{n,\infty} \text{ s.t. } x_n \to x\big\} \\ &\subset \big\{x \in \mathcal{X} : \exists x_n \in A_{n,\infty} \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\big\} \subset A. \end{aligned} \tag{2.3}$$

Proof If (2.3) holds, then

$$\begin{aligned} A_\infty &\subset \big\{x \in \mathcal{X}_\infty : \exists x_n \in A_{n,\infty} \text{ s.t. } x_n \to x\big\} \\ &\subset \big\{x \in \mathcal{X}_\infty : \exists x_n \in A_{n,\infty} \text{ s.t. } x \text{ is a cluster point of } (x_n)_{n\in\mathbb{N}}\big\} \subset A_\infty \end{aligned} \tag{2.4}$$

and conversely (2.4) implies (2.3), so the claim follows from Lemma 2.1 and the definition of d_c. Note that in (2.3) we cannot replace $A_{n,\infty}$ by A_n since some of the sets A_n may be empty. ∎

Let

$$\mathbb{S} := \overline{\mathbb{R}} \times \mathbb{R}. \tag{2.5}$$

If $\pi \in \Pi$ is a path with closed graph also denoted by π and filled graph $\overline{\pi}$, then we set

$$\pi^{\langle 2\rangle} := \big\{(z, z') \in \pi \times \pi : z \preceq z'\big\} \quad \text{and} \quad \overline{\pi}^{\langle 2\rangle} := \big\{(z, z') \in \overline{\pi} \times \overline{\pi} : z \preceq z'\big\}, \tag{2.6}$$

where $\preceq$ denotes the total order on π or $\overline{\pi}$, respectively. It is not hard to see that $\pi^{\langle 2\rangle}$ and $\overline{\pi}^{\langle 2\rangle}$ are closed subsets of $\mathbb{S}^2 = \mathbb{S} \times \mathbb{S}$, equipped with the product topology.

Lemma 2.3 (Convergence in the J1 and M1 topologies) *Let $\pi_n, \pi \in \Pi$. Then one has $\pi_n \to \pi$ in the J1 topology if and only if $\pi_n^{\langle 2\rangle} \to \pi^{\langle 2\rangle}$ in the local Hausdorff topology on $\mathrm{Cl}(\mathbb{S}^2)$. Similarly, $\pi_n \to \pi$ in the M1 topology if and only if $\overline{\pi}_n^{\langle 2\rangle} \to \overline{\pi}^{\langle 2\rangle}$ in the local Hausdorff topology.*

Proof Define π^* and $\overline{\pi}^*$ as in (1.5) and define $\pi^{*\langle 2\rangle}$ and $\overline{\pi}^{*\langle 2\rangle}$ as in (2.6) with $\pi \times \pi$ replaced by $\pi^* \times \pi^*$ and $\overline{\pi} \times \overline{\pi}$ replaced by $\overline{\pi}^* \times \overline{\pi}^*$. Then [FS23, Lemma 3.1] says that $\pi^{*\langle 2\rangle}$ and $\overline{\pi}^{*\langle 2\rangle}$ are compact subsets of $\mathbb{R}^2_\mathrm{c} \times \mathbb{R}^2_\mathrm{c}$, and [FS23, Thm 2.10] implies that $\pi_n \to \pi$ in the J1 topology if and only if $\pi_n^{*\langle 2\rangle} \to \pi^{*\langle 2\rangle}$ in the Hausdorff topology on $\mathcal{K}_+(\mathbb{R}^2_\mathrm{c} \times \mathbb{R}^2_\mathrm{c})$. Similarly, $\pi_n \to \pi$ in the M1 topology if and only if $\overline{\pi}_n^{*\langle 2\rangle} \to \overline{\pi}^{*\langle 2\rangle}$ in the Hausdorff topology on $\mathcal{K}_+(\mathbb{R}^2_\mathrm{c} \times \mathbb{R}^2_\mathrm{c})$.

We only prove the statement for the J1 topology. The same proof works for the M1 topology, with closed graphs replaced by filled graphs. For notational simplicity, we will assume that the closed graphs π_n are nonempty. If this is not the case, then in (2.7) below $\pi_n^{\langle 2\rangle}$ must be replaced by $\pi_n^{\langle 2\rangle} \cup \{\infty\}$ where ∞ is a point not included in $\mathbb{S}^2$. By Lemmas 2.1 and 2.2, we must show that

$$\begin{aligned}\pi^{\langle 2\rangle} &\subset \big\{(z,z') \in \mathbb{S}^2 : \exists (z_n,z'_n) \in \pi_n^{\langle 2\rangle} \text{ s.t. } (z_n,z'_n) \to (z,z')\big\} \\ &\subset \big\{(z,z') \in \mathbb{S}^2 : \exists (z_n,z'_n) \in \pi_n^{\langle 2\rangle} \text{ s.t. } (z,z') \text{ is a cluster point of } (z_n,z'_n)_{n\in\mathbb{N}}\big\} \subset \pi^{\langle 2\rangle}\end{aligned} \tag{2.7}$$

is equivalent to the same condition with $\pi^{\langle 2\rangle}$ and $\pi_n^{\langle 2\rangle}$ replaced by $\pi^{*\langle 2\rangle}$ and $\pi_n^{*\langle 2\rangle}$, and $\mathbb{S}$ replaced by $\mathbb{R}^2_{\rm c}$. Let us set $z_\pm := (*, \pm\infty)$. Then

$$\pi^{*\langle 2\rangle} = \pi^{\langle 2\rangle} \cup \big\{(z,z_+) : z \in \pi\big\} \cup \big\{(z_-,z) : z \in \pi\big\} \cup \big\{(z_-,z_-),(z_-,z_+),(z_+,z_+)\big\}. \tag{2.8}$$

We observe that (2.7) implies that

$$\begin{aligned}\pi &\subset \big\{z \in \mathbb{S} : \exists z_n \in \pi_n \text{ s.t. } z_n \to z\big\} \\ &\subset \big\{z \in \mathbb{S} : \exists z_n \in \pi_n \text{ s.t. } z \text{ is a cluster point of } (z_n)_{n\in\mathbb{N}}\big\} \subset \pi.\end{aligned} \tag{2.9}$$

Indeed, if $z \in \pi$, then $(z,z) \in \pi^{\langle 2\rangle}$ and hence by (2.7) there exist $(z_n,z'_n) \in \pi_n^{\langle 2\rangle}$ such that $(z_n,z'_n) \to (z,z)$, so in particular there exist $z_n \in \pi_n$ such that $z_n \to z$. Also, if $z_n \in \pi_n$ and a subsequence of $(z_n)_{n\in\mathbb{N}}$ converges to z, then $(z_n,z_n) \in \pi_n^{\langle 2\rangle}$ and a subsequence of $(z_n,z_n)_{n\in\mathbb{N}}$ converges to (z,z), so $(z,z) \in \pi^{\langle 2\rangle}$ by (2.7) and hence $z \in \pi$. Using (2.8) and (2.9), we see that (2.7) is equivalent to the same condition with $\pi^{\langle 2\rangle}$ and $\pi_n^{\langle 2\rangle}$ replaced by $\pi^{*\langle 2\rangle}$ and $\pi_n^{*\langle 2\rangle}$, and $\mathbb{S}$ replaced by $\mathbb{R}^2_{\rm c}$. ∎

As a corollary to the proof of Lemma 2.3 we obtain the following useful fact.

Lemma 2.4 (Convergence of graphs) *Let $\pi_n, \pi \in \Pi$. If $\pi_n \to \pi$ in the J1 topology, then the closed graphs π_n converge to π in the local Hausdorff topology on* Cl($\mathbb{S}$)*. Similarly, if $\pi_n \to \pi$ in the J1 topology, then the filled graphs $\overline{\pi}_n$ converge to $\overline{\pi}$ in the local Hausdorff topology on* Cl($\mathbb{S}$)*.*

Proof For the J1 topology this follows from the fact that (2.7) implies (2.9). The proof for the M1 topology is the same. ∎

2.2 Compactness criteria

Let $d_{\overline{\mathbb{R}}}$ be any metric generating the topology on the extended real line $\overline{\mathbb{R}}$. Let $d_{\overline{\mathbb{R}}}(x,A) := \inf_{y\in A} d(x,y)$ denote the distance of a point x to a subset $A \subset \overline{\mathbb{R}}$. For any path $\pi \in \Pi$, we define moduli of continuity by

$$\begin{aligned} m_{T,\delta}(\pi) &:= \sup\big\{d_{\overline{\mathbb{R}}}(x,y) : (x,y) \in \Delta^2_{T,\delta}(\pi)\big\}, \\ m^{\rm J}_{T,\delta}(\pi) &:= \sup\big\{d_{\overline{\mathbb{R}}}(y,\{x,z\}) : (x,y,z) \in \Delta^3_{T,\delta}(\pi)\big\}, \\ m^{\rm M}_{T,\delta}(\pi) &:= \sup\big\{d_{\overline{\mathbb{R}}}(y,[x,z]) : (x,y,z) \in \Delta^3_{T,\delta}(\pi)\big\}. \end{aligned} \tag{2.10}$$

Then [FS23, Thms 3.6 and 3.7] tell us the following.

Theorem 2.5 (Compactness criteria) *A set $\mathcal{A} \subset \Pi_{\rm c}$ is precompact if and only if*

$$\lim_{\delta\to 0} \sup_{\pi\in\mathcal{A}} m_{T,\delta}(\pi) = 0 \quad \forall T > 0. \tag{2.11}$$

A set $\mathcal{A} \subset \Pi$ is precompact with respect to the J1 topology if and only if

$$\lim_{\delta \to 0} \sup_{\pi \in \mathcal{A}} m^{\rm J}_{T,\delta}(\pi) = 0 \quad \forall T > 0. \tag{2.12}$$

A set $\mathcal{A} \subset \Pi$ is precompact with respect to the M1 topology if and only if

$$\lim_{\delta \to 0} \sup_{\pi \in \mathcal{A}} m^{\rm M}_{T,\delta}(\pi) = 0 \quad \forall T > 0. \tag{2.13}$$

We moreover mention [FS23, Lemma 2.8] which says the following.

Lemma 2.6 (Compactness in the Hausdorff topology) *Let $\mathcal{X}$ be a metrisable topological space and let $\mathcal{K}_+(\mathcal{X})$ be the space of nonempty compact subsets of $\mathcal{X}$, equipped with the Hausdorff topology. Then a set $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is precompact if and only if there exists a $C \in \mathcal{K}_+(\mathcal{X})$ such that $K \subset C$ for all $K \in \mathcal{A}$.*

Theorem 2.5 and Lemma 2.6 allow us to characterise the precompact subsets of $\mathcal{K}_+(\Pi_{\rm c})$, respectively of $\mathcal{K}_+(\Pi)$, with respect to the J1 and M1 topologies. We can use this to prove tightness criteria for the laws of random compact sets of paths. We start with continuous paths.

Theorem 2.7 (Tightness criterion for sets of continuous paths) *Let $(\mathcal{A}_\gamma)_{\gamma \in \Gamma}$ be a family of random variables with values in $\mathcal{K}_+(\Pi_{\rm c})$. Then the laws $(\mu_\gamma)_{\gamma \in \Gamma}$ with $\mu_\gamma := \mathbb{P}[\mathcal{A}_\gamma \in \cdot\,]$ are tight with respect to the topology on $\Pi_{\rm c}$ if and only if*

$$\lim_{\delta \to 0} \sup_{\gamma \in \Gamma} \mathbb{P}\big[\sup_{\pi \in \mathcal{A}_\gamma} m_{T,\delta}(\pi) > \varepsilon\big] = 0 \quad \forall T, \varepsilon > 0. \tag{2.14}$$

Proof Assume that the laws $(\mu_\gamma)_{\gamma \in \Gamma}$ are tight. Then for each $\eta > 0$, there exists a compact $\mathbf{C}_\eta \subset \mathcal{K}_+(\Pi_{\rm c})$ such that

$$\sup_{\gamma \in \Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \in \mathbf{C}_\eta\big] \geq 1 - \eta. \tag{2.15}$$

By Lemma 2.6, there exists a compact $\mathcal{C}_\eta \subset \Pi_{\rm c}$ such that $\mathcal{A} \in \mathbf{C}_\eta$ implies $\mathcal{A} \subset \mathcal{C}_\eta$. Thus, for each $\eta > 0$, there exists a compact $\mathcal{C}_\eta \subset \Pi_{\rm c}$ such that

$$\mathbb{P}\big[\mathcal{A}_\gamma \subset \mathcal{C}_\eta\big] \geq \mathbb{P}\big[\mathcal{A}_\gamma \in \mathbf{C}_\eta\big] \geq 1 - \eta \qquad (\gamma \in \Gamma). \tag{2.16}$$

By (2.11) of Theorem 2.5,

$$\lim_{\delta \to 0} w_{T,\eta}(\delta) = 0 \quad (T > 0,\ \gamma \in \Gamma) \quad \text{with} \quad w_{T,\eta}(\delta) := \sup_{\pi \in \mathcal{C}_\eta} m_{T,\delta}(\pi) \quad (T, \delta > 0,\ \gamma \in \Gamma). \tag{2.17}$$

Fix $T, \varepsilon > 0$. It follows that for each $\eta > 0$, there exists a $\delta > 0$ such that $w_{T,\eta}(\delta) \leq \varepsilon$ and hence, by (2.16) and the definition of $w_{T,\eta}(\delta)$,

$$\sup_{\gamma \in \Gamma} \mathbb{P}\big[\sup_{\pi \in \mathcal{A}_\gamma} m_{T,\delta}(\pi) > \varepsilon\big] \leq \sup_{\gamma \in \Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \not\subset \mathcal{C}_\eta\big] \leq \eta, \tag{2.18}$$

proving (2.14).

Assume, conversely, that (2.14) holds. Formula (2.14) implies that for each $\eta > 0$ and integers $T, n \geq 1$, there exists a $\delta_n(T, \eta) > 0$ such that

$$\sup_{\gamma \in \Gamma} \mathbb{P}\big[\sup_{\pi \in \mathcal{A}_\gamma} m_{T,\delta_n(T,\eta)}(\pi) > n^{-1}\big] \leq \eta 2^{-T-n}. \tag{2.19}$$

Summing over T and n, it follows that

$$\mathbb{P}\big[\exists T, n \geq 1 \text{ s.t. } \sup_{\pi \in \mathcal{A}_\gamma} m_{T,\delta_n(T,\eta)}(\pi) > n^{-1}\big] \leq \eta \qquad (\gamma \in \Gamma). \tag{2.20}$$

Setting

$$\mathcal{C}_\eta := \big\{\pi \in \Pi_{\rm c} : m_{T,\delta_n(T,\eta)}(\pi) \leq n^{-1} \ \forall T, n \geq 1\big\}, \tag{2.21}$$

our previous formula says that

$$\mathbb{P}\big[\mathcal{A}_\gamma \subset \mathcal{C}_\eta\big] \geq 1 - \eta \qquad (\gamma \in \Gamma). \tag{2.22}$$

We observe that $\delta \leq \delta_n(T,\eta)$ implies $m_{T,\delta}(\pi) \leq m_{T,\delta_n(T,\eta)}(\pi)$ and hence by the definition of $\mathcal{C}_\eta$

$$\sup_{\pi \in \mathcal{C}_\eta} m_{T,\delta}(\pi) \leq n^{-1} \qquad \forall T \geq 1, \ \delta \leq \delta_n(T,\eta). \tag{2.23}$$

It follows that

$$\limsup_{\delta \to 0} \sup_{\pi \in \mathcal{C}_\eta} m_{T,\delta}(\pi) \leq n^{-1} \quad \forall T, n \geq 1, \tag{2.24}$$

which by (2.11) of Theorem 2.5 implies that $\mathcal{C}_\eta$ is precompact. Letting $\overline{\mathcal{C}}_\eta$ denote its closure, we have for each $\eta > 0$ found a compact set $\overline{\mathcal{C}}_\eta \subset \Pi_{\rm c}$ such that

$$\sup_{\gamma \in \Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \subset \overline{\mathcal{C}}_\eta\big] \geq 1 - \eta. \tag{2.25}$$

By Lemma 2.6, this implies that the laws $(\mu_\gamma)_{\gamma \in \Gamma}$ are tight. ∎

In analogy with Theorem 2.7 we obtain the following tightness criteria for compact sets of cadlag paths.

Theorem 2.8 (Tightness criteria for sets of cadlag paths) *Let $(\mathcal{A}_\gamma)_{\gamma \in \Gamma}$ be a family of random variables with values in $\mathcal{K}_+(\Pi)$. Then the laws $(\mu_\gamma)_{\gamma \in \Gamma}$ with $\mu_\gamma := \mathbb{P}[\mathcal{A}_\gamma \in \cdot\,]$ are tight with respect to the J1 topology on Π if and only if*

$$\lim_{\delta \to 0} \sup_{\gamma \in \Gamma} \mathbb{P}\big[\sup_{\pi \in \mathcal{A}_\gamma} m^{\rm J}_{T,\delta}(\pi) \geq \varepsilon\big] = 0 \quad \forall T, \varepsilon > 0. \tag{2.26}$$

An analogue statement holds for the M1 topology, with $m^{\rm J}_{T,\delta}$ replaced by $m^{\rm M}_{T,\delta}$.

Proof The proof is completely the same as the proof of Theorem 2.7, using (2.12) and (2.13) of Theorem 2.5 instead of (2.11). ∎

2.3 Tightness criteria

In this subsection we derive Theorems 1.1 and 1.2 from Theorems 2.7 and 2.8. Fix $T, \delta > 0$. Recall the definitions of the sets $\Delta^2_{T,\delta}(\pi)$ and $\Delta^3_{T,\delta}(\pi)$ in (1.12) and (1.15). For each $\pi \in \Pi$, we set

$$\begin{aligned}
\Delta^{+}_{T,\delta}(\pi) &:= \big\{(x,y) \in \Delta^2_{T,\delta}(\pi) : x < y\big\},\\
\Delta^{+}_{T,\delta}(\pi) &:= \big\{(x,y) \in \Delta^2_{T,\delta}(\pi) : x > y\big\},\\
\Delta^{++}_{T,\delta}(\pi) &:= \big\{(x,y,z) \in \Delta^3_{T,\delta}(\pi) : x < y < z\big\},\\
\Delta^{+-}_{T,\delta}(\pi) &:= \big\{(x,y,z) \in \Delta^3_{T,\delta}(\pi) : x < y > z\big\},\\
\Delta^{-+}_{T,\delta}(\pi) &:= \big\{(x,y,z) \in \Delta^3_{T,\delta}(\pi) : x > y < z\big\},\\
\Delta^{--}_{T,\delta}(\pi) &:= \big\{(x,y,z) \in \Delta^3_{T,\delta}(\pi) : x > y > z\big\}.
\end{aligned} \tag{2.27}$$

For each $\eta > 0$ and $\star \in \{2, +, -\}$, we define

$$\mathcal{T}^{\star}_{T,\delta,\eta} := \big\{\pi \in \Pi : \exists (x,y) \in \Delta^{\star}_{T,\delta} \text{ s.t. } d_{\overline{\mathbb{R}}}(x,y) > \eta\big\}, \tag{2.28}$$

and for $\star \in \{3, ++, +-, -+, --\}$, we define

$$\mathcal{T}^{\star}_{T,\delta,\eta} := \big\{\pi \in \Pi : \exists (x,y,z) \in \Delta^{\star}_{T,\delta} \text{ s.t. } d_{\overline{\mathbb{R}}}(x,y), d_{\overline{\mathbb{R}}}(y,z) > \eta\big\}. \tag{2.29}$$

Finally, we set

$$\mathcal{T}^{\mathrm{J}}_{T,\delta,\eta} := \mathcal{T}^{++}_{T,\delta,\eta} \cup \mathcal{T}^{--}_{T,\delta,\eta} \quad \text{and} \quad \mathcal{T}^{\mathrm{M}}_{T,\delta,\eta} := \mathcal{T}^{+-}_{T,\delta,\eta} \cup \mathcal{T}^{-+}_{T,\delta,\eta}. \tag{2.30}$$

Lemma 2.9 (Alternative tightness criteria) *Condition (2.14) is equivalent to*

$$\lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \cap \mathcal{T}^{2}_{T,\delta,\eta} \neq \emptyset\big] = 0 \quad \forall T, \eta > 0. \tag{2.31}$$

Condition (2.26) is equivalent to

$$\lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \cap (\mathcal{T}^{\mathrm{J}}_{T,\delta,\eta} \cup \mathcal{T}^{\mathrm{M}}_{T,\delta,\eta}) \neq \emptyset\big] = 0 \quad \forall T, \eta > 0. \tag{2.32}$$

Condition (2.26) with $m^{\mathrm{J}}_{T,\delta}$ replaced by $m^{\mathrm{M}}_{T,\delta}$ is equivalent to

$$\lim_{\delta\to 0} \sup_{\gamma\in\Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \cap \mathcal{T}^{\mathrm{M}}_{T,\delta,\eta} \neq \emptyset\big] = 0 \quad \forall T, \eta > 0. \tag{2.33}$$

Proof We have $\sup_{\pi\in\mathcal{A}_\gamma} m_{T,\delta}(\pi) > \eta$ if and only if there exist $\pi \in \mathcal{A}_\gamma$ and $(x,y) \in \Delta^2_{T,\delta}(\pi)$ such that $d_{\overline{\mathbb{R}}}(x,y) > \eta$, so (2.14) is clearly equivalent to (2.31).

We have $\sup_{\pi\in\mathcal{A}_\gamma} m^{\mathrm{J}}_{T,\delta}(\pi) > \eta$ if and only if there exist $\pi \in \mathcal{A}_\gamma$ and $(x,y,z) \in \Delta^3_{T,\delta}(\pi)$ such that $d_{\overline{\mathbb{R}}}(x,y) \wedge d_{\overline{\mathbb{R}}}(y,z) > \eta$, so (2.26) is clearly equivalent to (2.32).

We have $\sup_{\pi\in\mathcal{A}_\gamma} m^{\mathrm{M}}_{T,\delta}(\pi) > \eta$ if and only if there exist $\pi \in \mathcal{A}_\gamma$ and $(x,y,z) \in \Delta^3_{T,\delta}(\pi)$ such that $d_{\overline{\mathbb{R}}}(y,[x,z]) > \eta$. Here $d_{\overline{\mathbb{R}}}(y,[x,z]) > \eta$ is equivalent to

$$d_{\overline{\mathbb{R}}}(x,y) \wedge d_{\overline{\mathbb{R}}}(y,z) > \eta \quad \text{and either } x,z < y \text{ or } x,z > y, \tag{2.34}$$

so (2.26) with $m^{\mathrm{J}}_{T,\delta}$ replaced by $m^{\mathrm{M}}_{T,\delta}$ is equivalent to (2.33). ∎

In what follows, it will be convenient to make a concrete choice for the metric $d_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$. We choose

$$d_{\overline{\mathbb{R}}}(x,y) := \big|\phi(x) - \phi(x)\big| \qquad (x,y \in \overline{\mathbb{R}}) \quad \text{with} \quad \phi(x) := \frac{x}{\sqrt{1+x^2}} \quad (x \in \mathbb{R}), \tag{2.35}$$

and $\phi(\pm\infty) := \pm 1$. We observe that

$$d_{\overline{\mathbb{R}}}(x,y) = \int_x^y (1+z^2)^{-3/2}\,\mathrm{d}z < y - x \qquad (x,y \in \mathbb{R},\ x < y). \tag{2.36}$$

For $\varepsilon > 0$, we choose $k_\pm(\varepsilon) \in \mathbb{Z}$ with $k_-(\varepsilon) < 0 < k_+(\varepsilon)$ such that

$$d_{\overline{\mathbb{R}}}\big(\pm\infty, k_\pm(\varepsilon)\varepsilon\big) < \varepsilon. \tag{2.37}$$

We need the following simple lemma.

Lemma 2.10 (Jumping over intervals) *For each* $\varepsilon > 0$ *and* $x, y \in \overline{\mathbb{R}}$ *such that* $x < y$ *and* $d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon$, *there exists a* $k \in \mathbb{Z}$ *with* $k_-(\varepsilon) \le k \le k_+(\varepsilon)-1$ *such that* $x \le k\varepsilon$ *and* $y \ge (k+1)\varepsilon$. *For each* $\varepsilon > 0$ *and* $x, y, z \in \overline{\mathbb{R}}$ *such that* $x < y < z$ *and* $d_{\overline{\mathbb{R}}}(x, y), d_{\overline{\mathbb{R}}}(y, z) > 2\varepsilon$, *there exists a* $k \in \mathbb{Z}$ *with* $k_-(\varepsilon) \le k \le k_+(\varepsilon) - 3$ *such that* $x \le k\varepsilon$, $(k + 1)\varepsilon \le y \le (k + 2)\varepsilon$, *and* $(k + 3)\varepsilon \le z$.

Proof Fix $\varepsilon > 0$. If $x, y \in \overline{\mathbb{R}}$ satisfy $x < y$ and $d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon$, then by (2.36) $y - x > 2\varepsilon$ and hence the set

$$K := \big\{k \in \mathbb{Z} : x \le k\varepsilon \text{ and } y \ge (k + 1)\varepsilon\big\} \tag{2.38}$$

is nonempty. We claim that if $k \in K$ satisfies $k < k_-(\varepsilon)$, then also $k_-(\varepsilon) \in K$. Indeed $k \in K$ and $k < k_-(\varepsilon)$ imply $x \le k\varepsilon < k_-(\varepsilon)\varepsilon$ while the observation that

$$d_{\overline{\mathbb{R}}}(-\infty, y) \ge d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon > d_{\overline{\mathbb{R}}}\big(-\infty, k_-(\varepsilon)\varepsilon\big) + \varepsilon > d_{\overline{\mathbb{R}}}\big(-\infty, (k_-(\varepsilon) + 1)\varepsilon\big) \tag{2.39}$$

implies $y \ge (k_-(\varepsilon)+1)\varepsilon$, completing the proof that $k_-(\varepsilon) \in K$. By the same argument, if $k \in K$ satisfies $k > k_+(\varepsilon) - 1$, then also $k_+(\varepsilon) - 1 \in K$ and the first claim of the lemma follows.

The proof of the second claim goes a bit differently. Assume that $x, y \in \overline{\mathbb{R}}$ satisfy $x < y$ and $d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon$. Choose $k \in \mathbb{Z}$ such that $(k + 1)\varepsilon \le y \le (k + 2)\varepsilon$. Since $d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon$, (2.36) tells us that $y - x > 2\varepsilon$ and hence $x \le k\varepsilon$. The same argument gives $(k + 3)\varepsilon \le z$. Since $y - (k + 1)\varepsilon \le \varepsilon$ (2.36) tells us that $d_{\overline{\mathbb{R}}}((k + 1)\varepsilon, y) < \varepsilon$. Combining this with the inequality $d_{\overline{\mathbb{R}}}(-\infty, y) \ge d_{\overline{\mathbb{R}}}(x, y) > 2\varepsilon$ we see that $d_{\overline{\mathbb{R}}}(-\infty, (k + 1)\varepsilon) > \varepsilon$ and hence $k_-(\varepsilon) < k + 1$. The same argument gives $k + 2 < k_+(\varepsilon)$. ∎

We next compare sets of the form $\mathcal{T}^{\mathrm{x}}_{T,\delta,\eta}$ with the sets $\mathcal{S}^{\mathrm{x}}_{T,\delta,\eta,r}$ defined in Subsection 1.3.

Lemma 2.11 (Comparison of sets of paths) *For* $\star \in \{+, -, +-, -+\}$ *one has*

$$\mathcal{S}^{\star}_{T,\delta,\varepsilon,r} \subset \mathcal{T}^{\star}_{T,\delta,\eta} \quad \textit{with} \quad \eta := d_{\overline{\mathbb{R}}}(r, r + \varepsilon) \qquad (T, \delta, \varepsilon > 0,\ r \in \mathbb{R}) \tag{2.40}$$

and

$$\mathcal{T}^{\star}_{T,\delta,2\varepsilon} \subset \bigcup_{k=k_-(\varepsilon)}^{k_+(\varepsilon)-1} \mathcal{S}^{\star}_{T,\delta,\varepsilon,k\varepsilon} \qquad (T, \delta, \varepsilon > 0). \tag{2.41}$$

For $\star \in \{++, --\}$ *one has*

$$\mathcal{S}^{\star}_{T,\delta,\varepsilon,r} \subset \mathcal{T}^{\star}_{T,\delta,\eta} \quad \textit{with} \quad \eta := d_{\overline{\mathbb{R}}}(r, r + \varepsilon) \wedge d_{\overline{\mathbb{R}}}(r + 2\varepsilon, r + 3\varepsilon) \qquad (T, \delta, \varepsilon > 0,\ r \in \mathbb{R}) \tag{2.42}$$

and

$$\mathcal{T}^{\star}_{T,\delta,2\varepsilon} \subset \bigcup_{k=k_-(\varepsilon)}^{k_+(\varepsilon)-3} \mathcal{S}^{\star}_{T,\delta,\varepsilon,k\varepsilon} \qquad (T, \delta, \varepsilon > 0). \tag{2.43}$$

Proof We first prove (2.40). If $\pi \in \mathcal{S}^{+}_{T,\delta,\varepsilon,r}$, then there exist $(x, y) \in \Delta^2_{T,\delta}(\pi)$ with $x < y$, $x \le r$, and $r + \varepsilon \le y$. Then $d_{\overline{\mathbb{R}}}(x, y) \ge d_{\overline{\mathbb{R}}}(r, r + \varepsilon)$ and hence $\pi \in \mathcal{T}^{+}_{T,\delta,\eta}$ with $\eta := d_{\overline{\mathbb{R}}}(r, r + \varepsilon)$. This proves (2.40) for $\star = +$. The same argument with the roles of x and y reversed yields (2.40) for $\star = -$. The arguments for $\star = +-$ and $\star = -+$ are also very similar, except that there are now three points x, y, z of which x, z lie on one side of the interval $[r, r + \varepsilon]$ while y lies on the other side.

The proof of (2.42) is also similar. If $\pi \in \mathcal{S}^{+}_{T,\delta,\varepsilon,r}$, then there exist $(x, y, z) \in \Delta^3_{T,\delta}(\pi)$ with $x < y < z$, $x \le r$, $r + \varepsilon \le y \le r + 2\varepsilon$, and $r + 3\varepsilon \le z$. Defining η as in (2.42) we then have $d_{\overline{\mathbb{R}}}(x, y) \ge d_{\overline{\mathbb{R}}}(r, r + \varepsilon) \ge \eta$ and $d_{\overline{\mathbb{R}}}(y, z) \ge d_{\overline{\mathbb{R}}}(r + 2\varepsilon, r + 3\varepsilon) \ge \eta$ which implies that $\mathcal{T}^{\star}_{T,\delta,\eta}$. This proves (2.42) for $\star = ++$. The proof for $\star = --$ is the same with the roles of x and z reversed.

We next prove (2.41). If $\pi \in \mathcal{T}^{+}_{T,\delta,2\varepsilon}$ then there exist $(x,y) \in \Delta^2_{T,\delta}(\pi)$ with $x < y$ and $d_{\overline{\mathbb{R}}}(x,y) > 2\varepsilon$. By Lemma 2.10 there then exists a $k \in \mathbb{Z}$ with $k_-(\varepsilon) \le k < k_+(\varepsilon)$ such that $x \le k\varepsilon$ and $y \ge (k+1)\varepsilon$. Then $\pi \in \mathcal{S}^{+}_{T,\delta,\varepsilon,k\varepsilon}$. This proves (2.41) for $\star = +$. The same argument with the roles of x and y reversed yields (2.41) for $\star = -$. For $\star = +-$ we argue as follows. If $\pi \in \mathcal{T}^{+-}_{T,\delta,2\varepsilon}$ then there exist $(x,y,z) \in \Delta^3_{T,\delta}(\pi)$ with $x, z < y$ and $d_{\overline{\mathbb{R}}}(x,y), d_{\overline{\mathbb{R}}}(z,y) > 2\varepsilon$. Then $x \vee z < y$ and $d_{\overline{\mathbb{R}}}(x \vee z, y) > 2\varepsilon$, so by Lemma 2.10 there then exists a $k \in \mathbb{Z}$ with $k_-(\varepsilon) \le k < k_+(\varepsilon)$ such that $x \vee z \le k\varepsilon$ and $y \ge (k+1)\varepsilon$, proving that $\pi \in \mathcal{S}^{+-}_{T,\delta,\varepsilon,k\varepsilon}$. The proof of (2.41) for $\star = -+$ is the same, using $x \wedge z$ instead of $x \vee z$.

It remains to prove (2.43). If $\pi \in \mathcal{T}^{++}_{T,\delta,2\varepsilon}$ then there exist $(x,y,z) \in \Delta^3_{T,\delta}(\pi)$ with $x < y < z$ and $d_{\overline{\mathbb{R}}}(x,y), d_{\overline{\mathbb{R}}}(z,y) > 2\varepsilon$. By Lemma 2.10 there then exists a $k \in \mathbb{Z}$ with $k_-(\varepsilon) \le k \le k_+(\varepsilon) - 3$ such that $x \le k\varepsilon$, $(k+1)\varepsilon \le y \le (k+2)\varepsilon$, and $(k+3)\varepsilon \le z$. Then $\pi \in \mathcal{S}^{++}_{T,\delta,\varepsilon,k\varepsilon}$. This proves (2.43) for $\star = ++$. The argument for $\star = --$ is the same with the roles of x and z reversed. ∎

Proof of Theorem 1.1 By Theorem 2.7 and Lemma 2.9 it suffices to prove that (1.14) is equivalent to (2.31). By formula (2.40) of Lemma 2.11, for all $T, \delta, \varepsilon > 0$ and $r \in \mathbb{R}$,

$$\mathbb{P}\big[\mathcal{S}^2_{T,\delta,\varepsilon,r} \cap \mathcal{A}_\gamma \neq \emptyset\big] \le \mathbb{P}\big[\mathcal{T}^2_{T,\delta,\eta} \cap \mathcal{A}_\gamma \neq \emptyset\big] \quad \text{with} \quad \eta := d_{\overline{\mathbb{R}}}(r, r+\varepsilon) \qquad (T,\delta,\varepsilon > 0,\ r \in \mathbb{R}), \tag{2.44}$$

from which we see that (2.31) implies (1.14). By formula (2.41) of Lemma 2.11, for all $T, \delta, \varepsilon > 0$,

$$\mathbb{P}\big[\mathcal{T}^2_{T,\delta,2\varepsilon} \cap \mathcal{A}_\gamma \neq \emptyset\big] \le \sum_{k=k_-(\varepsilon)}^{k_+(\varepsilon)-1} \mathbb{P}\big[\mathcal{S}^2_{T,\delta,\varepsilon,k\varepsilon} \cap \mathcal{A}_\gamma \neq \emptyset\big], \tag{2.45}$$

from which we see that (1.14) implies (2.31). ∎

Proof of Theorem 1.2 By Theorem 2.8 and Lemma 2.9 it suffices to prove that (1.18) (i) is equivalent to

$$\lim_{\delta \to 0} \sup_{\gamma \in \Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \cap \mathcal{T}^{\mathrm{M}}_{T,\delta,\eta} \neq \emptyset\big] = 0 \quad \forall T, \eta > 0 \tag{2.46}$$

and (1.18) (ii) is equivalent to

$$\lim_{\delta \to 0} \sup_{\gamma \in \Gamma} \mathbb{P}\big[\mathcal{A}_\gamma \cap \mathcal{T}^{\mathrm{J}}_{T,\delta,\eta} \neq \emptyset\big] = 0 \quad \forall T, \eta > 0. \tag{2.47}$$

By formulas (2.40) and (2.42) of Lemma 2.11, for each $T, \delta, \varepsilon > 0$ and $r \in \mathbb{R}$, there exists $\eta, \eta' > 0$ such that

$$\mathbb{P}\big[\mathcal{S}^{\mathrm{M}}_{T,\delta,\varepsilon,r} \cap \mathcal{A}_\gamma \neq \emptyset\big] \le \mathbb{P}\big[\mathcal{T}^{\mathrm{M}}_{T,\delta,\eta} \cap \mathcal{A}_\gamma \neq \emptyset\big] \quad \text{and} \quad \mathbb{P}\big[\mathcal{S}^{\mathrm{J}}_{T,\delta,\varepsilon,r} \cap \mathcal{A}_\gamma \neq \emptyset\big] \le \mathbb{P}\big[\mathcal{T}^{\mathrm{J}}_{T,\delta,\eta'} \cap \mathcal{A}_\gamma \neq \emptyset\big], \tag{2.48}$$

from which we see that (2.46) implies (1.18) (i) and (2.47) implies (1.18) (ii). The converse implications follow from formulas (2.41) and (2.43) of Lemma 2.11 which tell us that

$$\begin{aligned} \mathbb{P}\big[\mathcal{T}^{\mathrm{M}}_{T,\delta,\eta} \cap \mathcal{A}_\gamma \neq \emptyset\big] &\le \sum_{k=k_-(\varepsilon)}^{k_+(\varepsilon)-1} \mathbb{P}\big[\mathcal{S}^{\mathrm{M}}_{T,\delta,\varepsilon,k\varepsilon} \cap \mathcal{A}_\gamma \neq \emptyset\big], \\ \mathbb{P}\big[\mathcal{T}^{\mathrm{J}}_{T,\delta,\eta} \cap \mathcal{A}_\gamma \neq \emptyset\big] &\le \sum_{k=k_-(\varepsilon)}^{k_+(\varepsilon)-3} \mathbb{P}\big[\mathcal{S}^{\mathrm{J}}_{T,\delta,\varepsilon,k\varepsilon} \cap \mathcal{A}_\gamma \neq \emptyset\big]. \end{aligned} \tag{2.49}$$

∎

2.4 Noncrossing paths

In this subsection we prepare for the proof of Theorem 1.3. For $\pi_1, \pi_2 \in \Pi^{|}$, we say that π_1 *crosses* π_2 if $\pi_1 \not\trianglelefteq \pi_2$ and $\pi_2 \not\trianglelefteq \pi_1$, i.e., if the set $\{\pi_1, \pi_2\}$ is not noncrossing in the sense defined in Subsection 1.4. We say that π_1 *collides* with π_2 *at time* t if $t \in I_{\pi_1} \cap I_{\pi_2}$ and $\pi_1(t\pm), \pi_2(t\mp) < \pi_1(t\mp), \pi_2(t\pm)$, where the sign $\pm$ can be either $+$ or $-$ and $\mp$ is the opposite sign. In words, this says that at time t the paths π_1 and π_2 jump over some interval in opposite directions. It is not hard to see that if π_1 collides with π_2 at some time t, then π_1 crosses π_2. We let I° denote the interior of a closed real interval and let $\partial I := I \backslash I^\circ$ denote the set of its finite boundary points. We will prove the following result. Note that this shows in particular that for bi-infinite paths, the noncrossing property is preserved under limits.

Proposition 2.12 (Crossing in the limit) *Assume that $\pi_1^n, \pi_2^n \in \Pi^{|}$ satisfy $\pi_i^n \to \pi_i$ as $n \to \infty$ $(i = 1, 2)$ in the M1 topology for some $\pi_1, \pi_2 \in \Pi^{|}$, and that π_1^n does not cross π_2^n for any n. Then precisely one of the following statements must hold:*

(i) *π_1 does not cross π_2,*

(ii) *π_1 collides with π_2 at some time $t \in \partial I_{\pi_1} \cup \partial I_{\pi_2}$.*

Proposition 2.12 extends [FS24, Lemma 3.4.9]. It is not hard to guess the statement of Proposition 2.12 after scribbling a few figures on a piece of paper, but giving a precise proof is a bit more work. Our proof follows a clear strategy but is nevertheless quite long, so we challenge the reader to find a shorter one.

To prepare for the proof of Proposition 2.12, we start by giving a more direct characterisation of the relation $\pi_1 \trianglelefteq \pi_2$. The *split real line* is the set $\mathbb{R}_{\mathfrak{s}}$ consisting of all words of the form $t\star$ with $t \in \mathbb{R}$ and $\star \in \{-, +\}$. For each $\pi \in \Pi^{|}$, we define $I_\pi^{\mathfrak{s}} \subset \mathbb{R}_{\mathfrak{s}}$ by

$$I_\pi^{\mathfrak{s}} := \big\{ t\star : t \in I_\pi,\ \star \in \{-, +\},\ t\star \neq s-, u+ \big\}, \tag{2.50}$$

and letting $s := \inf I_\pi$ and $u := \sup I_\pi$ denote the starting time and final time of π, which may be $\pm\infty$, we define

$$\begin{aligned} I_\pi^{\mathrm{l}} &:= I_\pi^{\mathfrak{s}} \cup \big\{ s- : s \in \mathbb{R},\ \pi(s+) < \pi(s-) \big\} \cup \big\{ u+ : u \in \mathbb{R},\ \pi(u-) < \pi(u+) \big\}, \\ I_\pi^{\mathrm{r}} &:= I_\pi^{\mathfrak{s}} \cup \big\{ s- : s \in \mathbb{R},\ \pi(s-) < \pi(s+) \big\} \cup \big\{ u+ : u \in \mathbb{R},\ \pi(u+) < \pi(u-) \big\}, \end{aligned} \tag{2.51}$$

i.e., we include $s-$ in I_π^{l} only if $s \in \mathbb{R}$ and $\pi(s+) < \pi(s-)$, and so on. Finally, we set

$$\begin{aligned} L(\pi) &:= \big\{ (x, t\pm) \in \overline{\mathbb{R}} \times I_\pi^{\mathrm{l}} : x < \pi(t\pm) \big\}, \\ R(\pi) &:= \big\{ (x, t\pm) \in \overline{\mathbb{R}} \times I_\pi^{\mathrm{r}} : \pi(t\pm) < x \big\}. \end{aligned} \tag{2.52}$$

Note that these definitions simplify a lot for bi-infinite paths. The following lemma gives a more direct characterisation of the relation $\pi_1 \trianglelefteq \pi_2$.

Lemma 2.13 (Ordering of paths) *Two paths $\pi_1, \pi_2 \in \Pi^{|}$ satisfy $\pi_1 \trianglelefteq \pi_2$ if and only if $L(\pi_1) \cap R(\pi_2) = \emptyset$.*

Proof This is only a slight extension of [FS23, Lemma 3.3.2] but for completeness we give the proof. For bi-infinite paths $\pi_1, \pi_2 \in \Pi^{\updownarrow}$ the statement is trivial. It is also straightforward to check that if π' extends π, then $L(\pi) \subset L(\pi')$ and $R(\pi) \subset R(\pi')$. In view of this, the condition $L(\pi_1) \cap R(\pi_2) = \emptyset$ is clearly necessary for $\pi_1 \trianglelefteq \pi_2$. To prove sufficiency, we will prove the following statement:

(!) Assume that $\pi_1, \pi_2 \in \Pi^|$ satisfy $L(\pi_1) \cap R(\pi_2) = \emptyset$. Then there exist extensions π'_i of π_i $(i = 1, 2)$ such that $\pi'_1 \in \Pi^\uparrow$ and $L(\pi'_1) \cap R(\pi'_2) = \emptyset$.

By symmetry, it then follows that we can also exted π_2 to $+\infty$ and similarly both paths to $-\infty$ while preserving the property that $L(\pi_1) \cap R(\pi_2) = \emptyset$, which for bi-infinite paths trivially implies $\pi_1 \lhd \pi_2$. It therefore remains to prove (!). Let $s_i := \inf I_{\pi_i}$ and $u_i := \sup I_{\pi_i}$. The cases $u_1 = \pm\infty$ are trivial so without loss of generality we assume that $u_1 \in \mathbb{R}$. The cases $u_1 < s_2$ and $u_1 > u_2$ are trivial so without loss of generality we can assume that $s_2 \leq u_1 \leq u_2$. If $\pi_1(u_1+) \leq \pi_1(u_1-)$, then we can simply extend π_1 to the path that jumps to $-\infty$ at time u_1 and stays there, so we are done. If $\pi_1(u_1-) < \pi_1(u_1+)$, then $u_1+ \in I^|_\pi$ and hence $(x, u_1+) \in L(\pi)$ for all $x < \pi_1(u_1+)$. We now distinguish two cases: I. $u_1 < u_2$, and II. $u_1 = u_2 =: u$. In case I, the condition $L(\pi_1) \cap R(\pi_2) = \emptyset$ implies $\pi_1(u_1+) \leq \pi_2(u_1+)$, so we can extend π_1 to the path that jumps to $\pi_2(u_1+)$ at time u_1 and from then on stays equal to π_2. Case II is a bit more tricky. If $\pi_1(u+) \leq \pi_2(u+)$ we can proceed as in case I but it may also happen that $\pi_2(u+) < \pi_1(u+)$. If this happens, however, then the condition $L(\pi_1) \cap R(\pi_2) = \emptyset$ forces $\pi_2(u-) \leq \pi_2(u+)$ so in this case we can first extend π_2 to a path π'_2 satisfying $\pi_1(u+) \leq \pi_2(u+)$ and then again proceed as in case I. ∎

We can now also give a more direct characterisation of when two paths cross each other.

Lemma 2.14 (Crossing of paths) *Let $\pi_1, \pi_2 \in \Pi^|$. Then π_1 crosses π_2 if and only if $L(\pi_1) \cap R(\pi_2) \neq \emptyset$ and $L(\pi_2) \cap R(\pi_1) \neq \emptyset$.*

Proof Immediate from Lemma 2.13. ∎

We next start to investigate how crossing behaves under limits. For any $\pi \in \Pi^|$, we let π° denote the path whose domain is the closure of I°_π and that is equal to π on I°_π and does not jump at its finite boundary points. In other words, if I_π is an interval of positive length, then π° is obtained from π by removing jumps at finite boundary points, and if I_π consists of a single point or is empty, then π° is the trivial path with $I_{\pi^\circ} = \emptyset$. Recall from (2.5) that $\mathbb{S} := \overline{\mathbb{R}} \times \mathbb{R}$. For any $\pi \in \Pi^|$ we write

$$\begin{aligned} L^\circ(\pi) &:= \big\{(x,t) \in \mathbb{S} : t \in I^\circ_\pi,\ x < \pi(t-) \wedge \pi(t+)\big\}, \quad L^{\mathrm{c}}(\pi) := \mathbb{S} \backslash L^\circ(\pi), \\ \overline{L}(\pi) &:= \big\{(x,t) \in \mathbb{S} : t \in I_\pi,\ x \leq \pi(t-) \vee \pi(t+)\big\}. \end{aligned} \tag{2.53}$$

We define $R^\circ(\pi), R^{\mathrm{c}}(\pi)$, and $\overline{R}(\pi)$ analogously, with $x < \pi(t-) \wedge \pi(t+)$ replaced by $\pi(t-) \vee \pi(t+) < x$ etc. These sets look a lot like the sets $L(\pi)$ and $R(\pi)$ from (2.52), but they are better behaved under limits. Note that $L^\circ(\pi)$ is open while $L^{\mathrm{c}}(\pi)$ and $\overline{L}(\pi)$ are closed. Note also that $L^\circ(\pi) = L^\circ(\pi^\circ)$ and $\overline{L}(\pi^\circ)$ is the closure of $L^\circ(\pi)$.

Lemma 2.15 (Areas left of a path) *Assume that $\pi_n \in \Pi^|$ satisfy $\pi_n \to \pi$ as $n \to \infty$ in the M1 topology for some $\pi \in \Pi^|$. Then $\overline{L}(\pi_n) \to \overline{L}(\pi)$ and $L^{\mathrm{c}}(\pi_n) \to L^{\mathrm{c}}(\pi)$ in the local Hausdorff topology on* $\mathrm{Cl}(\mathbb{S})$.

Proof Assume that $\pi_n \to \pi$ in the M1 topology and the filled graphs $\overline{\pi}_n$ are not empty. Then by Lemma 2.4 $\overline{\pi}_n \to \overline{\pi}$ in the local Hausdorff topology on $\mathrm{Cl}(\mathbb{S})$, which by Lemma 2.2 means that:

(i) for each $z \in \overline{\pi}$ there exist $z_n \in \overline{\pi}_n$ such that $z_n \to z$,

(ii) if $z_n \in \overline{\pi}_n$ have a cluster point z, then $z \in \overline{\pi}$.

It is not hard to see that moreover $I_{\pi_n} \to I_\pi$ in the local Hausdorff topology on $\mathrm{Cl}(\mathbb{R})$.

We will only prove that $L^{\rm c}(\pi_n) \to L^{\rm c}(\pi)$. The proof that $\overline{L}(\pi_n) \to \overline{L}(\pi)$ is similar, but easier. We assume without loss of generality that $\overline{\pi}_n \neq \emptyset$ for all n. We note that

$$L^{\rm c}(\pi) = \big\{(x,t) \in \mathbb{S} : t \notin I^\circ_\pi\big\} \cup \big\{(x,t) \in \mathbb{S} : t \in I^\circ_\pi \text{ and } \pi(t-) \wedge \pi(t+) \leq x\big\}. \tag{2.54}$$

By Lemma 2.2, we have to show that:

(i)' for each $z \in L^{\rm c}(\pi)$ there exist $z_n \in L^{\rm c}(\pi_n)$ such that $z_n \to z$,

(ii)' if $z_n \in L^{\rm c}(\pi_n)$ have a cluster point z, then $z \in L^{\rm c}(\pi)$.

To prove (i)', assume that $(x,t) \in L^{\rm c}(\pi)$. Then either 1. $t \notin I^\circ_\pi$ or 2. $t \in I^\circ_\pi$ and $\pi(t-) \wedge \pi(t+) \leq x$. In case 1, we can use that $I_{\pi_n} \to I_\pi$ to choose $t_n \notin I^\circ_{\pi_n}$ such that $t_n \to t$. Now $(x,t_n) \in L^{\rm c}(\pi_n)$ satisfy $(x,t_n) \to (x,t)$. In case 2, we set $y := \pi(t-) \wedge \pi(t+)$. Then $(y,t) \in \overline{\pi}$ and hence by (i) there exist $(y_n,t_n) \in \overline{\pi}_n$ such that $(y_n,t_n) \to (y,t)$. Since $y \leq x$ it follows that $L^{\rm c}(\pi) \ni (y_n \vee x, t_n) \to (x,t)$.

To prove (ii)', assume that $(x_n,t_n) \in L^{\rm c}(\pi_n)$ have a cluster point (x,t). By going to a subsequence, we may assume that $(x_n,t_n) \to (x,t)$ and either 1. $t_n \notin I^\circ_{\pi_n}$ for all n or 2. $t_n \in I^\circ_{\pi_n}$ and $\pi_n(t_n-) \wedge \pi_n(t_n+) \leq x_n$ for all n. In case 1 by the fact that $I_{\pi_n} \to I_\pi$ we have $t = \lim_{n\to\infty} t_n \notin I^\circ_\pi$ so $(x,t) \in L^{\rm c}(\pi)$ and we are done. Case 2 is trivial if $t \notin I^\circ_\pi$ so without loss of generality we assume $t \in I^\circ_\pi$. Then $t_n \in I^\circ_\pi$ for n large enough so by going to a further subsequence if necessary, we can assume that $t_n \in I^\circ_\pi$ for all n and $y_n := \pi_n(t_n-) \wedge \pi_n(t_n+)$ converge to a limit $y \in \overline{\mathbb{R}}$. Then $\overline{\pi}_n \ni (y_n,t_n) \to (y,t)$ so by (i) $(y,t) \in \overline{\pi}$ and hence $\pi(t-) \wedge \pi(t+) \leq y$. Since $y_n \leq x_n$ we have $y \leq x$ so we conclude that $(x,t) \in L^{\rm c}(\pi)$ and we are done. ∎

We next need a lemma that characterises $\pi_1 \lhd \pi_2$ in terms of the sets $L^\circ, \overline{L}, R^\circ$, and $\overline{R}$. Below, for any real interval I with $s := \inf I$ and $u := \sup I$, we set

$$\partial^- I := \{s\} \cap \mathbb{R} \quad \text{and} \quad \partial^+ I := \{u\} \cap \mathbb{R}, \tag{2.55}$$

i.e., $\partial^- I$ and $\partial^+ I$ are the sets of finite lower and upper boundary points of I.

Lemma 2.16 (Ordering of non-colliding paths) *Assume that $\pi_1, \pi_2 \in \Pi^|$. Then one has $\pi_1 \lhd \pi_2$ if and only if the following conditions are satisfied:*

(i) $L^\circ(\pi_1) \cap \overline{R}(\pi_2) = \emptyset$,

(ii) $\overline{L}(\pi_1) \cap R^\circ(\pi_2) = \emptyset$,

(iii) *there does not exist a $t \in \partial^\pm I_{\pi_1} \cup \partial^\mp I_{\pi_2}$ with $t \in I_{\pi_1} \cap I_{\pi_2}$ such that $\pi_1(t\pm), \pi_2(t\mp) < \pi_1(t\mp), \pi_2(t\pm)$.*

Proof Recall that π° is the path π with jumps at finite boundary points of the domain removed. We start by proving that the following conditions are equivalent:

$$\text{(a) } \pi_1^\circ \not\lhd \pi_2^\circ, \quad \text{(b) } L^\circ(\pi_1) \cap R^\circ(\pi_2) \neq \emptyset \quad \text{(c) } L^\circ(\pi_1) \cap \overline{R}(\pi_2^\circ) \neq \emptyset. \tag{2.56}$$

By Lemma 2.13 and the definitions of $\pi^\circ, L(\pi), L^\circ(\pi), \overline{L}(\pi)$ etcetera, we see that (a)–(c) are equivalent to, respectively,

(a)' $\pi_2(t\star) < \pi_1(t\star)$ for some $t\star \in I^{\mathfrak{s}}_{\pi_1} \cap I^{\mathfrak{s}}_{\pi_2}$,

(b)' $\pi_2(t-) \wedge \pi_2(t+) < \pi_1(t-) \vee \pi_1(t+)$ for some $t \in I^\circ_{\pi_1} \cap I^\circ_{\pi_2}$, (c)' $\pi_2(t-)\wedge\pi_2(t+)<\pi_1(t-)\wedge\pi_1(t+)$ for some $t\in I^\circ_{\pi_1}\cap I^\circ_{\pi_2}$.

The implication (c)'$\Rightarrow$(a)' is trivial and the implication (a)'$\Rightarrow$(b)' follows from the observation that each $t-\in I^{\mathfrak{s}}_{\pi_1}\cap I^{\mathfrak{s}}_{\pi_2}$ can be approximated from below by $t_n\in I^\circ_{\pi_1}\cap I^\circ_{\pi_2}$ and likewise each $t+\in I^{\mathfrak{s}}_{\pi_1}\cap I^{\mathfrak{s}}_{\pi_2}$ can be approximated from above. The implication (b)$\Rightarrow$(c), finally, follows from the fact that $\overline{R}(\pi_2^\circ)$ is the closure of $R^\circ(\pi_2)$, so all three conditions are equivalent.

For any $\pi\in\Pi^{|}$ with initial and final time $s:=\inf I_\pi$ and $u:=\sup I_\pi$, let us set (compare (2.51))

$$
\begin{aligned}
L^-(\pi)&:=\big\{(x,s-):s\in\mathbb{R},\ \pi(s+)<\pi(s-)\big\},\\
L^+(\pi)&:=\big\{(x,u+):s\in\mathbb{R},\ \pi(u-)<\pi(u+)\big\},\\
R^-(\pi)&:=\big\{(x,s-):s\in\mathbb{R},\ \pi(s-)<\pi(s+)\big\},\\
R^+(\pi)&:=\big\{(x,u+):s\in\mathbb{R},\ \pi(u+)<\pi(u-)\big\}.
\end{aligned}
\tag{2.57}
$$

Then, by Lemma 2.13, $\pi_1\not\lhd\pi_2$ is equivalent to

$$
\big(L^-(\pi)\cup L(\pi^\circ)\cup L^+(\pi)\big)\cap\big(R^-(\pi)\cup R(\pi^\circ)\cup R^+(\pi)\big)\neq\emptyset. \tag{2.58}
$$

We observe that (2.58) holds precisely if one of the following conditions is satisfied:

I. $L(\pi_1^\circ)\cap R(\pi_2^\circ)\neq\emptyset$,

II. $L(\pi_1^\circ)\cap R^\star(\pi_2)\neq\emptyset$ for some $\star\in\{-,+\}$,

III. $L^\star(\pi_1)\cap R(\pi_2^\circ)\neq\emptyset$ for some $\star\in\{-,+\}$,

IV. $L^\star(\pi_1)\cap R^*(\pi_2)\neq\emptyset$ for some $\star,*\in\{-,+\}$.

For any $\pi\in\Pi^{|}$ and $t\in\overline{\mathbb{R}}$, let π^t denote the path with domain $I_{\pi^t}:=I_\pi\cap\{t\}$ defined by $\pi^t(t\pm):=\pi(t\pm)$ if $t\in I_\pi$. We claim that the conditions I–IV are equivalent, respectively, to:

I'. $L^\circ(\pi_1)\cap R^\circ(\pi_2)\neq\emptyset$,

II'. $\exists t\in I^\circ_{\pi_1}\cap\partial^\mp I_{\pi_2}$ s.t. $L^\circ(\pi_1)\cap\overline{R}(\pi_2^t)\neq\emptyset$ or $\pi_1(t\pm),\pi_2(t\mp)<\pi_1(t\mp),\pi_2(t\pm)$,

III'. $\exists t\in\partial^\pm I_{\pi_1}\cap I^\circ_{\pi_2}$ s.t. $\overline{L}(\pi_1^t)\cap R^\circ(\pi_2)\neq\emptyset$ or $\pi_1(t\pm),\pi_2(t\mp)<\pi_1(t\mp),\pi_2(t\pm)$,

IV'. $\exists t\in\partial^\pm I_{\pi_1}\cap\partial^\mp I_{\pi_2}$ s.t. $\pi_1(t\pm),\pi_2(t\mp)<\pi_1(t\mp),\pi_2(t\pm)$.

The equivalence of I and I' has already been proved. Let us set $s_i:=\inf I_{\pi_i}$ and $u_i:=\sup I_{\pi_i}$ $(i=1,2)$. We observe that $L(\pi_1^\circ)\cap R^-(\pi_2)\neq\emptyset$ if and only if $s_2\in I^\circ_{\pi_1}$, $\pi_2(s_2-)<\pi_2(s_2+)$, and $\pi_2(s_2-)<\pi_1(s_2-)$. There are now two possibilities: if $\pi_2(s_2-)<\pi_1(s_2+)$, then $L^\circ(\pi_1)\cap\overline{R}(\pi_2^{s_2})\neq\emptyset$, and if $\pi_1(s_2+)\leq\pi_2(s_2-)$, then $\pi_1(s_2+),\pi_2(s_2-)<\pi_1(s_2-),\pi_2(s_2+)$. The argument is the same if $L(\pi_1^\circ)\cap R^-(\pi_2)\neq\emptyset$, except that now all signs are reversed so we arrive at the condition that $L^\circ(\pi_1)\cap\overline{R}(\pi_2^{u_2})\neq\emptyset$ or $\pi_1(u_2-),\pi_2(u_2+)<\pi_1(u_2+),\pi_2(u_2-)$. This proves the equivalence of II and II'. The equivalence of III and III' follows from the same argument, again with all signs reversed, and the proof of the equivalence of IV and IV' is similar.

To complete the proof, it now suffices to observe that $L^\circ(\pi_1)\cap\overline{R}(\pi_2)\neq\emptyset$ if and only if $L^\circ(\pi_1)\cap R^\circ(\pi_2)\neq\emptyset$ or there exists a $t\in I^\circ_{\pi_1}\cap\partial I_{\pi_2}$ such that $L^\circ(\pi_1)\cap\overline{R}(\pi_2^t)\neq\emptyset$, and the condition $L^\circ(\pi_1)\cap\overline{R}(\pi_2)\neq\emptyset$ can be rewritten similarly. ∎

The final ingredient in the proof of Proposition 2.12 is the following lemma.

Lemma 2.17 (Limits of ordered paths) *Assume that $\pi_1^n,\pi_2^n\in\Pi^{|}$ satisfy $\pi_1^n\lhd\pi_2^n$ for all n and that $\pi_i^n\to\pi_i$ as $n\to\infty$ $(i=1,2)$ in the M1 topology for some $\pi_1,\pi_2\in\Pi^{|}$. Then $L^\circ(\pi_1)\cap\overline{R}(\pi_2)=\emptyset$ and $\overline{L}(\pi_1)\cap R^\circ(\pi_2)=\emptyset$.*

Proof By symmetry, it suffices to prove that $L^\circ(\pi_1)\cap\overline{R}(\pi_2)=\emptyset$ or equivalently $\overline{R}(\pi_2)\subset L^c(\pi_1)$. Since $\pi_1^n \lhd \pi_2^n$ for all n, Lemma 2.16 tells us that $\overline{R}(\pi_2^n)\subset L^c(\pi_1^n)$ for all n, so the claim follows from Lemma 2.15, using Lemma 2.2. ∎

Proof of Proposition 2.12 Assume that $\pi_1^n,\pi_2^n\in\Pi^|$ satisfy $\pi_i^n\to\pi_i$ as $n\to\infty$ $(i=1,2)$ in the M1 topology for some $\pi_1,\pi_2\in\Pi^|$, and that π_1^n does not cross π_2^n for any n. It is clear from Lemma 2.14 that (i) and (ii) cannot hold simultaneously, so it suffices to prove that if π_1 does not collide with π_2 at any time $t\in\partial I_{\pi_1}\cup\partial I_{\pi_2}$, then π_1 does not cross π_2.

Since π_1^n does not cross π_2^n for any n, by going to a subsequence, we can assume that either $\pi_1^n \lhd \pi_2^n$ for all n or $\pi_2^n \lhd \pi_1^n$ for all n. By symmetry, we can without loss of generality assume that we are in the first case. Then Lemma 2.17 tells us that $L^\circ(\pi_1)\cap\overline{R}(\pi_2)=\emptyset$ and $\overline{L}(\pi_1)\cap R^\circ(\pi_2)=\emptyset$. By Lemma 2.16 and the fact that π_1 does not collide with π_2 at any time $t\in\partial I_{\pi_1}\cup\partial I_{\pi_2}$, it follows that $\pi_1\lhd\pi_2$, so π_1 does not cross π_2. ∎

2.5 Tightness of noncrossing sets

In this subsection we prove Theorem 1.3 and Lemma 1.4. We start by showing that (1.18) (i) implies (1.22).

Lemma 2.18 (Jumps in opposite directions) *Let $\mathcal{A}$ be a compact subset of $\Pi^|$, let $T,\delta,\varepsilon>0$ and $r\in\mathbb{R}$. Then*

$$\mathcal{S}^{\rm M}_{T,\delta,\varepsilon,r}\cap\mathcal{A}_\gamma\neq\emptyset\quad\textit{implies}\quad\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r}\cap(\mathcal{A}_\gamma\times\mathcal{A}_\gamma)\neq\emptyset. \tag{2.59}$$

Proof One has $\mathcal{S}^{+-}_{T,\delta,\varepsilon,r}\cap\mathcal{A}_\gamma\neq\emptyset$ if and only if:

(S) There exists a $\pi\in\mathcal{A}$ and $(x,s),(y,t),(z,u)\in\pi$ with $(x,s)\preceq(y,t)\preceq(z,u)$, $-T\leq s\leq t\leq u\leq T$, and $u-s\leq\delta$, such that $x,z\leq r$ and $r+\varepsilon\leq y$.

Similarly, one has $\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r}\cap(\mathcal{A}_\gamma\times\mathcal{A}_\gamma)\neq\emptyset$ if and only if:

(C) For $i=1,2$ there exists $\pi_i\in\mathcal{A}$ and $(x_i,s_i),(y_i,t_i)\in\pi_i$ with $(x_i,s_i)\preceq(y_i,t_i)$, $-T\leq s_i\leq t_i\leq T$, and $(t_1\vee t_2)-(s_1\wedge s_2)\leq\delta$, such that $x_1,y_2\leq r$ and $r+\varepsilon\leq y_1,x_2$.

If (S) holds, then setting $\pi_1=\pi_2:=\pi$, $(x_1,s_1):=(x,s)$, $(y_1,t_1)=(x_2,s_2):=(y,t)$, and $(y_2,t_2):=(z,u)$, we see that (C) holds. The same argument with the roles of π_1 and π_2 interchanged shows that $\mathcal{S}^{-+}_{T,\delta,\varepsilon,r}\cap\mathcal{A}_\gamma\neq\emptyset$ implies (C). ∎

Lemma 2.19 (Crossing in the limit) *Let $\Pi^|$ be equipped with the M1 topology and let $\mathcal{A}_n$ be noncrossing compact subsets of $\Pi^|$. Assume that $\mathcal{A}_n\to\mathcal{A}$ in the Hausdorff topology on $\mathcal{K}_+(\Pi)$. Then $\mathcal{A}\subset\Pi^|$. If $\mathcal{A}$ is noncrossing, then*

$$\forall T,\varepsilon>0,\ r\in\mathbb{R}\ \exists\delta>0,\ m\in\mathbb{N}\ \textit{s.t.}\ \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r}\cap(\mathcal{A}_n\times\mathcal{A}_n)=\emptyset\ \forall n\geq m. \tag{2.60}$$

On the other hand, if $\mathcal{A}$ is not noncrossing, then

$$\exists T,\varepsilon>0,\ r\in\mathbb{R}\ \textit{s.t.}\ \forall\delta>0\ \exists m\in\mathbb{N}\ \textit{s.t.}\ \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r}\cap(\mathcal{A}_n\times\mathcal{A}_n)\neq\emptyset\ \forall n\geq m. \tag{2.61}$$

Proof It is easy to see that $\Pi^|$ is a closed subset of Π. As a result, by grace of Lemma 2.1, $\mathcal{K}_+(\Pi^|)$ is a closed subset of $\mathcal{K}_+(\Pi)$. This proves that $\mathcal{A}\subset\Pi^|$.

Let us say that $\mathcal{A}$ *contains colliding paths* if there exist $\pi_1,\pi_2\in\mathcal{A}$ and $t\in\mathbb{R}$ such that π_1 collides with π_2 at time t. We will prove the following statements:

I. $\mathcal{A}$ is noncrossing if and only if $\mathcal{A}$ does not contain colliding paths.

II. If $\mathcal{A}$ does not contain colliding paths, then (2.60) holds.

III. If $\mathcal{A}$ contains colliding paths, then (2.61) holds.

We start by proving I. It is clear that if $\mathcal{A}$ contains colliding paths, then $\mathcal{A}$ is not noncrossing. To prove the converse, assume that $\mathcal{A}$ does not contain colliding paths and let $\pi_1, \pi_2 \in \mathcal{A}$. By Lemma 2.1, there exist $\pi_1^n, \pi_2^n \in \mathcal{A}_n$ such that $\pi_i^n \to \pi_i$ $(i = 1, 2)$. Since $\mathcal{A}_n$ is noncrossing, for each n, the path π_1^n does not cross π_2^n. By Proposition 2.12, it follows that either π_1 collides with π_2 or π_1 does not cross π_2. The first option is excluded by our assumption that $\mathcal{A}$ does not contain colliding paths, so we conclude that $\mathcal{A}$ is noncrossing.

We next prove II. We will show that if (2.60) does not hold, then $\mathcal{A}$ contains colliding paths. If (2.60) does not hold, then there exist $T, \varepsilon > 0$ and $r \in \mathbb{R}$ such that

$$\forall \delta > 0,\ m \in \mathbb{N}\ \exists n \geq m \text{ s.t. } \mathcal{C}^{\mathrm{M}}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset. \tag{2.62}$$

Fix $\delta_k > 0$ such that $\delta_k \to 0$. Then we can choose $n(k) \to \infty$ such that $\mathcal{C}^{\mathrm{M}}_{T,\delta_n,\varepsilon,r} \cap (\mathcal{A}_{n(k)} \times \mathcal{A}_{n(k)}) \neq \emptyset$ for each k. This implies that for $i = 1, 2$ there exists $\pi_i^k \in \mathcal{A}_{n(k)}$ and $(x_i^k, s_i^k), (y_i^k, t_i^k) \in \pi_i^k$ with $(x_i^k, s_i^k) \preceq (y_i^k, t_i^k)$, $-T \leq s_i^k \leq t_i^k \leq T$, and $(t_1^k \vee t_2^k) - (s_1^k \wedge s_2^k) \leq \delta_k$, such that $x_1^k, y_2^k \leq r$ and $r + \varepsilon \leq y_1^k, x_2^k$. By Lemma 2.1 and the fact that $\mathcal{A}_{n(k)} \to \mathcal{A}$, there exists a compact $\mathcal{C} \subset \Pi$ such that $\mathcal{A}_{n(k)} \subset \mathcal{C}$ for all k. In view of this, using also the compactness of $\overline{\mathbb{R}}$ and $[-T, T]$, by going to a subsequence if necessary, we can assume that as $k \to \infty$

$$\pi_i^k \to \pi_i,\ (x_i^k, s_i^k) \to (x_i, s_i), \quad (y_i^k, t_i^k) \to (y_i, t_i) \quad (i = 1, 2). \tag{2.63}$$

By Lemma 2.1 we have $\pi_1, \pi_2 \in \mathcal{A}$. Combining Lemma 2.1 with Lemma 2.3 we see that $(x_i, s_i), (y_i, t_i) \in \overline{\pi}_i$ with $(x_i, s_i) \preceq (y_i, t_i)$. Taking the limit in $x_1^k, y_2^k \leq r$ and $r + \varepsilon \leq y_1^k, x_2^k$ we obtain that $x_1, y_2 \leq r$ and $r + \varepsilon \leq y_1, x_2$. Finally, since $(t_1^k \vee t_2^k) - (s_1^k \wedge s_2^k) \leq \delta_k$, we must have $s_1 = s_2 = t_1 = t_2 =: t$ for some $t \in [-T, T]$. We have to be a bit careful since as a result of using the M1 topology we only know that $(x_i, s_i), (y_i, t_i)$ are elements of the filled graph $\overline{\pi}_i$, and not necessarily of the closed graph π_i. Nevertless, the properties we have proved are enough to conclude that π_1 collides with π_2 at time t.

It remains to prove III. Assume that $\pi_1, \pi_2 \in \mathcal{A}$ collide at time t. Then $t \in [-T/2, T/2]$ for T large enough and (possibly after interchanging the roles of π_1 and π_2) there exist x_i, y_i with $(x_i, t), (y_i, t) \in \overline{\pi}_i$ and $(x_i, t) \preceq (y_i, t)$ $(i = 1, 2)$ such that $x_1, y_2 \leq r - \varepsilon$ and $r + 2\varepsilon \leq y_1, x_2$ for some $\varepsilon > 0$ and $r \in \mathbb{R}$. By Lemma 2.1 there exist $\pi_1^n, \pi_2^n \in \mathcal{A}_n$ such that $\pi_i^n \to \pi_i$ $(i = 1, 2)$, so by Lemmas 2.1 and 2.3 there exist $(x_i^n, s_i^n), (y_i^n, t_i^n) \in \overline{\pi}_i^n$ with $(x_i^n, s_i^n) \preceq (y_i^n, t_i^n)$ such that $x_i^n \to x_i$ and $y_i^n \to y_i$ $(i = 1, 2)$. Then for each $\delta > 0$ one has $(t_1^n \vee t_2^n) - (s_1^n \wedge s_2^n) \leq \delta$ for k large enough. Moreover, for all k large enough $s_i^n, t_i^n \in [-T, T]$, $x_1^n, y_2^n \leq r$, and $r + \varepsilon \leq y_1^n, x_2^n$. By making x_1^n, y_2^n smaller if necessary and y_1^n, x_2^n larger if necessary, we can make sure that $(x_i^n, s_i^n), (y_i^n, t_i^n)$ are elements of the closed graph π_i^n and not just of the filled graph $\overline{\pi}_i^n$, while preserving all other properties mentioned above. Then

$$\forall \delta > 0\ \exists m \in \mathbb{N} \text{ s.t. } \mathcal{C}^{\mathrm{M}}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m, \tag{2.64}$$

which shows that (2.61) holds. ∎

Proof of Theorem 1.3 Set $\mathcal{M} := \{\mu_\gamma : \gamma \in \Gamma\}$. Let $\mathcal{M}_1(\mathcal{K}_+(\Pi))$ denote the space of probability measures on $\mathcal{K}_+(\Pi)$, equipped with the topology of weak convergence. We naturally view $\mathcal{M}_1(\mathcal{K}_{\mathrm{nc}}(\Pi^|))$ as a subset of $\mathcal{M}_1(\mathcal{K}_+(\Pi))$. Then $\mathcal{M}$ is precompact as a subset of $\mathcal{M}_1(\mathcal{K}_{\mathrm{nc}}(\Pi^|))$

if and only if $\mathcal{M}$ is precompact as a subset of $\mathcal{M}_1(\mathcal{K}_+(\Pi))$ and the closure of $\mathcal{M}$ is contained in $\mathcal{M}_1(\mathcal{K}_{\rm nc}(\Pi^{|}))$. Thus, by Prohorov's theorem, $\mathcal{M}$ is tight on $\mathcal{K}_{\rm nc}(\Pi^{|})$ if and only if $\mathcal{M}$ is tight on $\mathcal{K}_+(\Pi)$ and the closure of $\mathcal{M}$ is concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$. Therefore, by Theorem 1.2, it suffices to prove the following statements:

I. If (1.22) holds, then (1.18) (i) holds and the closure of $\mathcal{M}$ is concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$.

II. If (1.18) (i) holds and the closure of $\mathcal{M}$ is concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$, then (1.22) holds.

We first prove I. Assume that (1.22) holds. Then by Lemma 2.18 condition (1.18) (i) is satisfied, so it remains to show that the closure of $\mathcal{M}$ is concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$. Assume that $\mu_n \in \mathcal{M}$ converge weakly to a probability law μ on $\mathcal{K}_+(\Pi)$. By Skorohod's representation theorem [EK86, Cor 3.1.6 and Thm 3.1.8], we can couple random variables $\mathcal{A}_n, \mathcal{A}$ with laws μ_n, μ such that $\mathcal{A}_n \to \mathcal{A}$ a.s. We need to show that $\mathcal{A} \in \mathcal{K}_{\rm nc}(\Pi^{|})$ a.s. By Lemma 2.19 we have $\mathcal{A} \subset \Pi^{|}$ a.s. Therefore, by condition (2.61) of Lemma 2.19, to prove that $\mathcal{A} \in \mathcal{K}_{\rm nc}(\Pi^{|})$ a.s., it suffices to show that

$$\mathbb{P}\big[\exists T,\varepsilon > 0,\ r \in \mathbb{R} \text{ s.t. } \forall \delta > 0,\ \exists m \in \mathbb{N} \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m\big] = 0. \tag{2.65}$$

It suffices to check the condition (2.61) for countably many values of T, ε, and r only: in particular, we can take $T = N$, $\varepsilon = 1/n$, and $r = k/(3n)$ with N, n positive integers and $k \in \mathbb{Z}$. In view of this, it suffices to show that

$$\mathbb{P}\big[\forall \delta > 0,\ \exists m \in \mathbb{N} \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m\big] = 0 \quad \forall T,\varepsilon > 0,\ r \in \mathbb{R}. \tag{2.66}$$

Since $\delta' \leq \delta$ implies $\mathcal{C}^{\rm M}_{T,\delta',\varepsilon,r} \subset \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r}$, we can rewrite our previous formula as

$$\lim_{\delta\to 0} \mathbb{P}\big[\exists m \in \mathbb{N} \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m\big] = 0 \quad \forall T,\varepsilon > 0,\ r \in \mathbb{R}. \tag{2.67}$$

To see that (1.22) implies (2.67), we estimate

$$\begin{aligned}\mathbb{P}\big[\exists m \in \mathbb{N} \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m\big] &= \lim_{m\to\infty} \mathbb{P}\big[\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\ \forall n \geq m\big] \\ \leq \limsup_{m\to\infty} \mathbb{P}\big[\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_m \times \mathcal{A}_m) \neq \emptyset\big] &\leq \sup_{m\in\mathbb{N}} \mathbb{P}\big[\mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_m \times \mathcal{A}_m) \neq \emptyset\big].\end{aligned} \tag{2.68}$$

Inserting this into the left-hand side of (2.67) and using (1.22) we see that the right-hand side of (2.67) is zero, completing the proof of I.

It remains to prove II. We will prove that if (1.18) (i) holds and (1.22) fails, then there exist μ in the closure of $\mathcal{M}$ that are not concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$. Fix $\delta_n > 0$ such that $\delta_n \to 0$. Since (1.22) does not hold, there exist $T, \varepsilon, \eta > 0$ and $r \in \mathbb{R}$ such that for each $n \geq 1$ we can find a random variable $\mathcal{A}_n$ with law $\mu_n \in \mathcal{M}$ such that

$$\mathbb{P}\big[\mathcal{C}^{\rm M}_{T,\delta_n,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\big] \geq \eta. \tag{2.69}$$

Since (1.18) (i) holds, by Theorem 1.2, by going to a subsequence if necessary, we can assume that $\mu_n \Rightarrow \mu$ for some probability law μ on $\mathcal{K}_+(\Pi)$. Let $\mathcal{A}$ have law μ. By Skorohod's representation theorem, we can couple $\mathcal{A}_n, \mathcal{A}$ in such a way that $\mathcal{A}_n \to \mathcal{A}$ a.s. Then (2.69) implies that

$$\mathbb{P}\big[\forall m \in \mathbb{N}\ \exists n \geq m \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta_n,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\big] \geq \eta, \tag{2.70}$$

and hence

$$\mathbb{P}\big[\exists T,\varepsilon > 0,\ r \in \mathbb{R} \text{ s.t. } \forall \delta > 0,\ m \in \mathbb{N}\ \exists n \geq m \text{ s.t. } \mathcal{C}^{\rm M}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\big] \geq \eta. \tag{2.71}$$

By condition (2.60) of Lemma 2.19, this shows that

$$\mathbb{P}\big[\mathcal{A} \text{ is noncrossing}\big] \leq 1 - \eta, \tag{2.72}$$

so μ is not concentrated on $\mathcal{K}_{\rm nc}(\Pi^{|})$. ∎

Proof of Lemma 1.4 Assume that $\mathcal{A}_n \in \mathcal{K}_{\rm nc}(\Pi^{\updownarrow})$ and that $\mathcal{A}_n \to \mathcal{A}$ for some $\mathcal{A} \in \mathcal{K}_+(\Pi)$. Then by Lemma 2.1, for each $\pi \in \mathcal{A}$, there exist $\pi_n \in \mathcal{A}_n$ such that $\pi_n \to \pi$. It is easy to see that $\Pi^{\updownarrow}$ is a closed subset of Π, so $\pi \in \Pi^{\updownarrow}$ for all $\pi \in \mathcal{A}$. It remains to show that $\mathcal{A}$ is noncrossing. Let $\pi_1, \pi_2 \in \mathcal{A}$. By Lemma 2.1, there exist $\pi_1^n, \pi_2^n \in \mathcal{A}_n$ such that $\pi_i^n \to \pi_i$ $(i = 1, 2)$. Since $\mathcal{A}_n$ is noncrossing, for each n, the path π_1^n does not cross π_2^n. By Proposition 2.12, it follows that π_1 does not cross π_2, proving that $\mathcal{A}$ is noncrossing. ∎

3 Applications

Throughout Section 3 we will use a piece of notation that is in common usage for the Brownian web, and related objects, but that we have not so far introduced. For $\pi \in \Pi^{\uparrow}$ we write σ_π for the initial time of π i.e. if π has time domain $[t, \infty)$ then $\sigma_\pi = t$.

3.1 Weaves

In this section we prove a result that specialises Theorem 1.3. A *weave*, introduced and studied in [FS24], is a random compact subset $\mathcal{A} \subseteq \Pi^{\uparrow}$ that is non-crossing and for which, with probability one, for all $z \in \mathbb{R}^2$ there exists $\pi \in \mathcal{A}$ such that $z \in \overline{\pi}$ i.e. the paths cover space-time. We consider weaves with a law that is invariant under deterministic translations of space-time. In this setting, we show that tightness of the motion of a single particle in the sense of the *classical* Skorohod space $D_{[0,\infty)}(\mathbb{R})$ (as defined in e.g. [EK86, Bil99, Whi02]) essentially controls tightness of the whole system in $\mathcal{K}(\Pi^{\uparrow})$. We show also that translation invariance can be replaced by a more technical condition that is uniform in space and time.

Before stating the result we must note a subtle technical point. Let (X_n) be a sequence of $D_{[0,\infty)}(\mathbb{R})$ valued random variables. Although $D_{[0,\infty)}(\mathbb{R})$ is a subspace of $\Pi^{\uparrow}$, tightness of (X_n) in $\Pi^{\uparrow}$ (in the sense of Theorem 1.2) does not imply tightness of (X_n) in $D_{[0,\infty)}(\mathbb{R})$ (in the classical sense). The distinction arises because for $\pi \in D_{[0,\infty)}(\mathbb{R})$ we must have $\pi(0-) = \pi(0+)$ whereas Π permits jumps at time 0, more precisely

$$D_{[0,\infty)}(\mathbb{R}) = \big\{\pi \in \Pi^{\uparrow} : \sigma_\pi = 0,\ \pi(0-) = \pi(0+)\big\}. \tag{3.1}$$

Tightness conditions for $D_{[0,\infty)}(\mathbb{R})$ (as found in [EK86, Bil99, Whi02] and suchlike) must therefore enforce that limit points do not jump at time 0, whereas tightness conditions (such as Theorems 1.2 and 1.3) for the larger space $\Pi^{\uparrow}$ need not do so. For single paths the precise connection, which is a straightforward consequence of [FS23, Thm 3.10], is as follows.

Lemma 3.1 (Tightness on $D_{[0,\infty)}(\mathbb{R})$) *Let (X_n) be a sequence of random variables with values in $D_{[0,\infty)}(\mathbb{R})$. Then their laws are tight on $D_{[0,\infty)}(\mathbb{R})$ under J1 (resp. M1) if and only if these laws are tight on Π under J1 (resp. M1) and additionally, for all $\varepsilon > 0$ we have $\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \mathbb{P}\big[\sup_{t\in[0,\delta]} |X_n(t)| \geq \varepsilon\big] = 0$.*

We now recount some important facts from [FS24, Section 2.4] concerning weaves. Let $\mathcal{A}$ be a weave. Then, except for z within a (deterministic) null set $R_{\mathcal{A}} \subseteq \mathbb{R}^2$, for each $z \in \mathbb{R}^2$ there exists an almost surely unique path $\pi_z \in \Pi^{\uparrow}$ that begins at z and does not cross $\mathcal{A}$. We say that π_z is the *one-particle motion* of $\mathcal{A}$ from z.

We say that a weave $\mathcal{A}$ is *homogeneous* if the law of $\mathcal{A}$ is invariant under deterministic translations of space and time. In this case the set $R_{\mathcal{A}}$ is empty: for each $z \in \mathbb{R}^2$ there exists an almost surely unique random path $\pi_z \in \Pi^{\uparrow}$ that begins at z and does not cross $\mathcal{A}$. Moreover the law of the path

$$t \mapsto \pi_{(x,s)}(s+t) - x \qquad \text{for } t \in [0,\infty) \tag{3.2}$$

does not depend on $z = (x, s)$. The law of this process might include a jump at time $t = 0$, but if it does not do so then (3.2) is a $D_{[0,\infty)}(\mathbb{R})$ valued random variable. We refer to this law as the one-particle motion of $\mathcal{A}$.

For a weave $\mathcal{A}$, let $\mathcal{F}_t$ be the σ-field generated by $\{\pi(s)\,;\, \pi \in \mathcal{A},\, \sigma_\pi \leq s \leq t\}$. We say that $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is the generated filtration of $\mathcal{A}$.

Theorem 3.2 (Tightness criteria for homogeneous weaves) *Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of weaves, with generated filtrations $(\mathcal{F}^n_t)$. Assume the following:*

(i) *For each $n \in \mathbb{N}$, $\mathcal{A}_n$ is homogeneous.*

(ii) *For each $n \in \mathbb{N}$, let X_n be the one-particle motion of $\mathcal{A}_n$ from the origin. Suppose that X_n takes values in $D_{[0,\infty)}(\mathbb{R})$ and is strongly Markov with respect to the filtration $(\mathcal{F}^n_t)_{t\in[0,\infty)}$.*

(iii) *The laws of (X_n) are tight on $D_{[0,\infty)}(\mathbb{R})$, under J1 or M1.*

Then the laws of $(\mathcal{A}_n)$ are tight on $\mathcal{K}(\Pi^{\uparrow})$ under M1 and all weak limit points are non-crossing.

As a corollary to the proof of Theorem 3.2 we will obtain the following more general, but more technical result.

Corollary 3.3 (Tightness criteria for inhomogeneous weaves) *In the setting of Theorem 3.2, the same conclusion can be drawn if instead of conditions (i)-(iii) we have that for each $n \in \mathbb{N}$ and $z \in \mathbb{R}^2$ there exists a random path $\pi^n_z \in \Pi^{\uparrow}$, which does not cross $\mathcal{A}_n$, such that:*

(iv) *π^n_z is strongly Markov with respect to $(\mathcal{F}^n_t)$.*

(v) *For all $\varepsilon > 0$, $x \in \mathbb{R}$ and $T \in (0,\infty)$,*

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \sup_{z\in\{x\}\times[-T,T]} \mathbb{P}\left[\sup_{s\in[0,\delta]} |\pi^n_z(s^n_z + s) - \pi^n_z(s^n_z)| \geq \varepsilon\right] = 0. \tag{3.3}$$

Proof The proof of Theorem 3.2 and Corollary 3.3, which we give together, will take up the remainder of Section 3.1. Let us first handle the connection between the two results, namely we show that (i)-(iii) implies both (iv) and (v). To see this take π^n_z to be the one-particle motion of $\mathcal{A}_n$ from z, which by (i) and our general comments above is well defined for all $z \in \mathbb{R}^2$, with a distribution that does not depend on z. Condition (iv) then follows from (ii) and translation invariance. Condition (v) follows from (iii) and Lemma 3.1. It therefore suffices to establish the conclusion of Theorem 3.2 under conditions (iv) and (v), which we assume hereon.

Let $T, \delta \in (0,\infty)$ and $r \in \mathbb{R}$. For each $n \in \mathbb{N}$ we define a sequence of paths π^n_m and stopping times τ^n_m. Set $\tau^n_0 = -T$ and then for $m \geq 1$,

$$\pi^n_m = \pi^n_{(r+\varepsilon,\tau_{m-1})}, \qquad \tau^n_m = \inf\{t \geq \tau^n_{m-1}\,;\, \pi^n_m \leq r \text{ or } \pi^n_m \geq r + 2\varepsilon\}.$$

In words, the path π^n_m begins at time τ^n_{m-1} at spatial location $r + \varepsilon$. The time τ^n_m occurs when π^n_m exits the region $[r, r + 2\varepsilon]$, upon which π^n_{m+1} is born at space-time location $(r + \varepsilon, \tau^n_m)$. If

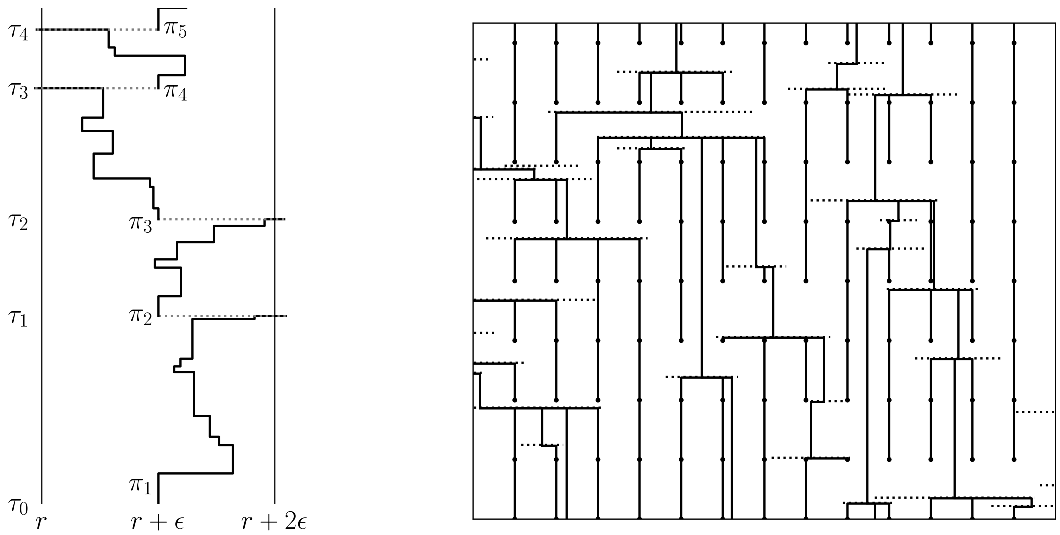

Figure 3.1: In both images, time runs upwards and space is on the horizontal axis.
On the left: A depiction of the sequences $(\tau^n_m)_{m\in\mathbb{N}}$ and $(\pi^n_m)_{m\in\mathbb{N}}$, with the superscript n dropped. Note that if a path π is to cross $[r, r+2\varepsilon]$ then it must pass through at least one of the grey dotted sections at time τ^n_m. Moreover, any such crossing must be from left to right if the corresponding π^n_m exits $[r, r+2\varepsilon]$ via the right boundary, or must be from from right to left if the corresponding π^n_m exits $[r, r+2\varepsilon]$ via the left boundary. The event E^n_δ ensures that the dotted sections are separated from one another in time by at least δ, which prevents the event $\mathcal{C}^{\mathrm{M}}_{T,\delta,2\varepsilon,0} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset$ from occurring.
On the right: Part of a Poisson tree, with one path beginning at each black dot and a square border encasing the space-time window displayed. Events $(x, t, r) \in P_n$ are shown as dotted sections of the form $[x-r, x+r] \times \{t\}$. The realization of the underlying Poisson process P_n has been chosen to make a clear picture, rather than to demonstrate heavy tailed behaviour.

τ^n_m is infinite, then the inductive definition terminates and the sequences defined are taken to be finite.

Define also the events

$$E^n_{\delta,m} = \left\{\tau^n_m > \tau^n_{m-1} + \delta\right\}, \qquad E^n_\delta = \bigcap_{m=1}^{M_n} E^n_{\delta,m}$$

where $M_n = \inf\{m \in \mathbb{N}\,;\, \tau_m \geq T\}$. We claim that, in the notation of (1.22),

$$\left\{\mathcal{C}^{\mathrm{M}}_{T,\delta,2\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\right\} \subseteq \Omega \setminus E^n_\delta. \tag{3.4}$$

The reasoning behind equation (3.4) is explained in Figure 3.1, which depicts the sequences (τ^n_m) and (π^n_m). In order to check (1.22) we must therefore check that the event E^n_δ occurs with high probability, in a sense matching the limits appearing in (1.22).

By (3.3) there exists $\delta_0 > 0$ such that

$$\inf_n \inf_{z\in\{r+\varepsilon\}\times[-T,T]} \mathbb{P}\left[\sup_{s\in[0,\delta_0]} \left|\pi^n_z(\sigma_{\pi^n_z} + s) - \pi^n_z(\sigma_{\pi^n_z})\right| < \varepsilon\right] \geq \frac{1}{2},$$

which implies that, for all $m \in \mathbb{N}$, $\inf_n \mathbb{P}\left[\tau^n_m - \tau^n_{m-1} \geq \delta_0\right] \geq \frac{1}{2}$. The strong Markov property from condition (iv) gives that, for fixed n, the variables $(\tau^n_m - \tau^n_{m-1})_{m\in\mathbb{N}}$ are independent. Noting that $\tau^n_m - \tau^n_0 = \sum_{i=1}^m \tau^n_m - \tau^n_{m-1}$, it follows that τ^n_m is, for all n, stochastically bounded below by $-T + \sum_1^m T_i$ where $\mathbb{P}[T_i \geq \delta_0] \geq \frac{1}{2}$. Hence M_n is bounded above by a negative binomial distribution with parameters $N = \lceil 2T/\delta_0 \rceil$ (the number of 'successes') and $p = \frac{1}{2}$. Noting that these parameters are independent of n, it follows that

$$\text{for all } \kappa > 0 \text{ there exists } K \in \mathbb{N} \text{ such that } \sup_{n\in\mathbb{N}} \mathbb{P}[M_n \geq K] \leq \kappa. \tag{3.5}$$

Let $\kappa > 0$ and take $K \in \mathbb{N}$ as in (3.5), then

$$\begin{aligned} \mathbb{P}[\Omega \setminus E^n_\delta] &\leq \mathbb{P}\left[\bigcup_{m=1}^{M_n} \Omega \setminus E^n_{\delta,m} \text{ and } M_n < K\right] + \mathbb{P}[M_n \geq K] \\ &\leq \sum_{m=1}^{K} \mathbb{P}[\Omega \setminus E^n_{\delta,m}] + \kappa \\ &\leq \sum_{m=1}^{K} \mathbb{P}\left[\sup_{s\in[0,\delta]} \left|\pi^n_z(\sigma_{\pi^n_z} + s) - \pi^n_z(\sigma_{\pi^n_z})\right| \geq \varepsilon\right] + \kappa. \end{aligned} \tag{3.6}$$

The first line of the above follows by elementary set algebra and conditioning. The second line follows from (3.5) and the third from the definition of $E^n_{\delta,m}$. Applying (v) to let $\delta \to 0$, we obtain that $\limsup_{\delta\to 0} \sup_{n\in\mathbb{N}} \mathbb{P}[\Omega \setminus E^n_\delta] \leq \kappa$, and using that $\kappa > 0$ was arbitrary we obtain $\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} \mathbb{P}[\Omega \setminus E^n_\delta] = 0$. In view of (3.4) and Theorem 1.3, this completes the proof. ∎

3.2 Heavy tailed Poisson trees

In this section we use Theorem 3.2 to show tightness of a specific particle system, under its natural rescaling. We treat a heavy tailed version of the Poisson trees considered by [FFW05, EFS17] and others, in which the motion of a single particle is within the natural domain of attraction of an α-stable process, where $\alpha \in (0, 2)$. For brevity we show only tightness. We do not attempt to characterize the limit, which will be a system of highly correlated non-crossing α-stable processes. In the case $\alpha = 2$, after a diffusive rescaling the particles become coalescing Brownian motions that are independent before coalescence; under suitable conditions the limit in this case is known to be the Brownian web. However in the α-stable case one should expect that particles in the limit are dependent even before coalescence.

Fix $\alpha \in (0, 2)$ and let μ be a finite measure on $(0,\infty)$ such that

$$\lim_{R\to\infty} R^\alpha \int_R^\infty r\,\mu(dr) \;\in (0,\infty), \tag{3.7}$$

for example $\mu(dr) = (1 \wedge r^{-\alpha-2})\,dr$. For each $n \in \mathbb{N}$ let P_n denote a Poisson point process in $\mathbb{R} \times \mathbb{R} \times (0,\infty)$ with intensity measure

$$n^{1/\alpha}\,dx \otimes n\,dt \otimes \mu_n(dr) \tag{3.8}$$

where $\mu_n(A) = \mu\left(n^{1/\alpha} A\right)$.

Given a point $z = (x,t) \in \mathbb{R}^2$ we define a cadlag path π^n_z with time domain $[t,\infty)$ and initial point (x,t), by specifying that π^n_z remains constant except at values of t for which $\pi^n_z(t-) \in [x-r, x+r]$ for some $(x,t,r) \in P_n$; at such times the path jumps and $\pi^n_z(t+) = x$. We will show below that π^n_z is a compound Poisson process, which by symmetry has zero mean.

We define

$$W_n = \{\pi^n_z\,;\, z \in \mathbb{R}^2\}$$

and let $\mathcal{W}_n$ be the closure of W_n in $\Pi^\uparrow$. See Figure 3.1 for a graphical depiction of W_n and P_n. Equation (3.8) corresponds to a space-time rescaling in which, at stage n, we speed up time by a factor n and compress space by a factor $n^{1/\alpha}$. In particular, for fixed but arbitrary $z \in \mathbb{R}^2$ and $n \in \mathbb{N}$, for $s \in (0,\infty)$ the processes

$$s \mapsto \pi^n_z(s) \text{ and } s \mapsto n^{-1/\alpha}\pi^1_z(ns) \text{ have the same law.} \tag{3.9}$$

The remainder of the present section will apply Theorem 3.2 to the sequence $(\mathcal{W}_n)$. We must first show that $\mathcal{W}_n$ is a weave. Then we must check conditions (i)-(iii) of Theorem 3.2; conditions (i) and (ii) are essentially immediate and condition (iii) will come from showing that the limiting one-particle motion is α-stable.

Remark 3.4 For arbitrary $Z_n \subseteq \mathbb{R}^2$ we have that $A_n = \{\pi^n_z\,;\, z \in Z_n\} \subseteq W_n \subseteq \mathcal{W}_n$. Setting $\mathcal{A}_n$ to be the closure of A_n, tightness of the sequence $(\mathcal{A}_n)$ follows immediately from that of $(\mathcal{W}_n)$. For example, we might use paths begun on rescalings of the square lattice $\mathbb{Z}^2_n = \{(x,t) \in \mathbb{R}^2\,;\, n^{1/\alpha}x \in \mathbb{Z},\ nt \in \mathbb{Z}\}$ as depicted in Figure 3.1.

We begin with the one-particle motion. More precisely, we will show that the process $X_n(s) = \pi^{(n)}_{(0,0)}(s)$ is a compound Poisson process. In view of (3.9) it suffices to consider $n = 1$. Let us note at the outset that condition (3.7) combined with the requirement that μ be a finite measure implies that $\int_0^\infty r\,\mu(dr) < \infty$.

We refer to each $(x,t,r) \in P_n$ as an event and to the space-time region $[x-r, x+r] \times \{t\}$ as being affected by the event. We similarly say that when $X_n(t-) \in [x-r, x+r]$, the path X_n is affected by the event (x,t,r). Taking $n = 1$, if the current location of $X_1 = \pi^{(1)}_{(0,0)}$ is $y \in \mathbb{R}$ then it is affected by events at rate

$$\int_0^\infty \int_{\mathbb{R}} \mathbb{1}_{\{y \in [x-r,x+r]\}}\, dx\, \mu(dr) = 2\int_0^\infty r\,\mu(dr) \tag{3.10}$$

which is finite. Upon being affected by an event, the resulting jump changes the spatial location of X_n by (addition of) a random variable J for which

$$\begin{aligned}
\mathbb{P}[J \geq R] &= \frac{1}{K}\int_0^\infty \int_{\mathbb{R}} \mathbb{1}_{\{y \in [x-r,x-R]\}}\, dx\, \mu(dr) \\
&= \frac{1}{K}\int_0^\infty (r-R) \vee 0\ \mu(dr) \\
&= \frac{1}{K}\int_R^\infty r\,\mu(dr),
\end{aligned} \tag{3.11}$$

where $R \in (0,\infty)$ and $K = 2\int_0^\infty r\,\mu(dr)$. Note that by symmetry for $R \geq 0$ we have $\mathbb{P}[J \geq R] = \mathbb{P}[J \leq -R]$, so (3.11) characterizes the distribution of J.

Proposition 3.5 (Tightness of heavy tailed Poisson trees) *The laws of $(\mathcal{W}_n)$ are tight on $\mathcal{K}(\Pi^\uparrow)$ under M1 and all weak limit points are non-crossing.*

Proof We first justify that $\mathcal{W}_n$ is a weave, for each $n \in \mathbb{N}$. With $n \in \mathbb{N}$ fixed, it is clear that $\mathcal{W}_n$ is a subset of $\Pi^\uparrow$ and that each $z \in \mathbb{R}^2$ satisfies $z \in \overline{\pi}$ for some $\pi \in \mathcal{W}_n$. To see that $\mathcal{W}_n$ is a weave, it remains only to show that $\mathcal{W}_n$ is almost surely compact and non-crossing, for each $n \in \mathbb{N}$.

A similar calculation to (3.10) shows that for each $L, T \in (0,\infty)$ there exists, almost surely, a random $\delta > 0$ (depending on L, T and n) such that the regions $[x-r, x+r] \times [t-\delta, t+\delta]$ are disjoint for all $(x,t,r) \in P_n$ with $(x,t) \in [-L,L] \times [-T,T]$. Note that paths in W_n remain constant (in space) outside of such regions. With this in hand it is easily seen via Theorem 2.5

that $\mathcal{W}_n = \overline{W_n}$ is almost surely compact, under both J1 and M1. Moreover, noting that W_n is non-crossing, Proposition 2.12 gives that if two paths $\pi, \pi' \in \mathcal{W}_n$ were to cross, then they must collide at time $\sigma_\pi \vee \sigma_{\pi'}$. However, this eventuality cannot occur due to $\delta > 0$. Hence $\mathcal{W}_n$ is almost surely non-crossing.

To complete the proof we will apply Theorem 3.2 to the sequence $(\mathcal{W}_n)_{n\in\mathbb{N}}$, for which we must check conditions (i)-(iii) of that theorem. Conditions (i) and (ii) follow immediately from standard independence properties and translation-invariance of Poisson point processes, so it remains only to check condition (iii).

We have seen above that the one-particle motion of $\mathcal{W}_n$ from the origin is a compound Poisson process with jump rate (3.10) and jump distribution J characterized by (3.11). Theorem 4.5.2 of [Whi02] gives that a *discrete* time random walk with i.i.d. increments having distribution J is within the normal domain of attraction of an α-stable process, with time rescaled by a factor n and space rescaled by $n^{-1/\alpha}$. Theorem 4.5.3 of [Whi02] gives that such a walk converges in law, as a cadlag process under the J1 (and hence also M1) topology, to an α-stable process. For brevity we do not calculate the scale parameter of the limiting α-stable process here but it may be found via formulae therein; by symmetry the skewness and shift parameters are both zero. Noting (3.10), we are in fact concerned with a random walk in continuous time i.e. a compound Poisson process. The two cases differ only by a strictly increasing piecewise linear time change, a time change which in the limit converges almost surely, in the locally uniform sense, to a linear time change. Thus, by Theorem 13.2.2 of [Whi02], the same results apply to the sequence $n \mapsto X_n(\cdot)$. This establishes condition (iii) of Theorem 3.2, which completes the proof. ∎

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