Seven lectures on percolation

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1 Introduction of the models

1.1 Unoriented bond percolation

Let $\mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbb{Z}\ \forall i = 1, \ldots, d\}$ $(d \geq 1)$ be the d-dimensional integer lattice. We may view $\mathbb{Z}^d$ as an infinite (unoriented) graph, the vertices of which are points $x \in \mathbb{Z}^d$, with two vertices x, y connected by an edge if and only if $|x - y| = 1$. (Such vertices are sometimes called *nearest neighbors*.) We denote the set of edges by $\mathcal{E}(\mathbb{Z}^d) := \{\{x, y\} : |x - y| = 1\}$. In the context of percolation, the edges are often called *bonds*, and the vertices *sites*. Imagine that each edge (or bond) $e \in \mathcal{E}(\mathbb{Z}^d)$ can be in two states: with probability p, such an edge is *open*, and with the remaining probability $1 - p$ it is *closed*, independently for all edges, where $p \in [0, 1]$ is some fixed parameter.

In dimension $d = 3$, we can think of this as a simple model for a porous stone, where the vertices represent small cavities that may or may not be connected to neighboring cavities. We are interested in the question whether water (or oil) can flow through such a stone. Here, we postulate that water can flow from a site x to a site y if and only if x and y are connected by a *path of open edges*, i.e., if there exist $z_0, \ldots, z_n$ such that $z_0 = x$, $z_n = y$, and z_{k-1} and z_k are connected by an open edge for each $k = 1, \ldots, n$. If this is the case, then we will write $x \leftrightarrow y$ (where, by definition, $x \leftrightarrow x$ for all x).

We call $C(x) := \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$ the *open cluster at x*. We can think of $C(x)$ as the sites that will be wetted if we introduce water to the stone at the site x. We are naturally interested in the size of $C(x)$. In particular, we may ask whether $C(x)$ can be infinite. Because of translation invariance, it suffices to consider the open cluster at the origin $x = 0$. We define the *percolation probability* $\theta(p) = \theta_d(p)$ (depending on the *percolation parameter* p and the dimension d) as

$$\theta(p) := P_p\big[|C(0)| = \infty\big] \qquad (p \in [0, 1]). \tag{1.1}$$

We start by noting that in dimension $d = 1$ the function θ is trivial, since in this case $\theta(p) = 0$ for all $p < 1$ and $\theta(1) = 1$. Therefore, unless stated otherwise, *from now on we assume that* $d \geq 2$. The function θ has been studied a lot both by theoretical physicists and mathematicians since the 1950-ies. As usual in these matters, the physicists, not hampered by the restrictions of rigorous proof, got much further than the mathematicians. As a result, the behavior of θ is now probably well-understood from the point of view of most theoretical physicists, but still contains a wealth of open problems for mathematicians. In this section we give an overview of the most important facts about θ that are known or believed to hold, while in the next sections, we will start proving a very few of them.

It is known that θ is nondecreasing as a function of p, and that there exists a *critical parameter* $p_\mathrm{c} = p_\mathrm{c}(d)$ (depending on the dimension) with $0 < p_\mathrm{c} < 1$, such that $\theta(p) = 0$ for $p < p_\mathrm{c}$ but $\theta(p) > 0$ for $p > p_\mathrm{c}$. The function θ is strictly increasing and right-continuous on $[p_\mathrm{c}, 1]$, and continuous on $(p_\mathrm{c}, 1]$. It is believed that θ is also left-continuous at p_c (i.e., $\theta(p_\mathrm{c}) = 0$), but this has been proved only in dimension $d = 2$ and in sufficiently high dimensions ($d \geq 19$ was the state of the art in 1999; the methods require at least $d > 6$. This was proved in [HS90, HS94], see also [Gri99].) It is believed that θ is concave on $[p_\mathrm{c}, 1]$, but no progress has been made on that conjecture for 50 years.

Theoretical physicists have made very precise claims about the behavior of θ near the critical point p_c. Let a_n, b_n be strictly positive constants with $\lim_{n\to\infty} b_n = 0$ or ∞. We will write

$$\begin{aligned} a_n \sim b_n \qquad &\text{if } \lim_{n\to\infty} \frac{a_n}{b_n} = 1, \\ a_n \approx b_n \qquad &\text{if } \lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1. \end{aligned} \tag{1.2}$$

Note that $a_n \sim b_n$ iff $a_n = (1+\varepsilon_n)b_n$ and $a_n \approx b_n$ iff $a_n = b_n^{1+\varepsilon_n}$, where $\varepsilon_n \to 0$. It is not hard to see that $a_n \sim cb_n$ for some $0 < c < \infty$ implies $a_n \approx b_n$, but the converse is not true; for example, $n \log n \approx n$. If f and g are functions of some continuous parameter t, then we define in a similar way $f(t) \sim g(t)$ and $f(t) \approx g(t)$, as t is sent to some limiting value. With these conventions, it is widely believed that

$$\theta(p) \approx (p - p_c)^\beta \quad \text{as } p \downarrow p_c, \tag{1.3}$$

where $\beta = \beta(d) > 0$ is a *critical exponent* that depends on the dimension. Here $\beta(2) = \frac{5}{36}$ and $\beta(d) = 1$ for $d \geq 6$. The values of $\beta(3), \ldots, \beta(5)$ interpolate between these values. They are known numerically, but no explicit formula is known. Some authors claim that in all dimensions except $d = 6$, formula (1.3) can be strengthened to $\theta(p) \sim c(p - p_c)^\beta$ for some $0 < c < \infty$. In dimension 6, it is believed that the asymptotic formula for $\theta(p)$ contains a logarithmic factor. Little of this is rigorous, except in high dimensions ($d \geq 19$). There has been a lot of recent progress on understanding the case $d = 2$, which involves conformal field theory, but as far as I am aware of this has not yet yielded a proof of (1.3).

Exercise Let $a_n, a'_n, b_n, b'_n > 0$ and assume either $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = 0$ or $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \infty$. Prove that $a_n \sim b_n$ and $a'_n \sim b'_n$ implies $a_n a'_n \sim b_n b'_n$. Prove that moreover $a_n \approx b_n$ and $a'_n \approx b'_n$ implies $a_n a'_n \approx b_n b'_n$. How is the situation if $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} b'_n = \infty$?

1.2 Oriented bond percolation

Instead of viewing $\mathbb{Z}^d$ as an unoriented graph, as we did in the last section, we can also make $\mathbb{Z}^d$ into a oriented graph, in the following special way. Introduce an order on $\mathbb{Z}^d$ by writing $x \leq y$ if $x_i \leq y_i$ for all $i = 1, \ldots, d$. Note that if $x, y \in \mathbb{Z}^d$ and $|x - y| = 1$, then either $x \leq y$ or $y \leq x$. Replacing the set of nearest neighbor edges $\mathcal{E}(\mathbb{Z}^d)$ by a set of arrows $\mathcal{A}(\mathbb{Z}^d)$ which point from the lower site in a pair of nearest neighbors to the higher, we obtain an oriented graph whose vertices are $\mathbb{Z}^d$.

In analogy with the unoriented case, we let each arrow, independently of the others, be open with probability p, and closed with the remaining probability. We say that there is an *oriented* (or *directed*) *path of open arrows* from a site x to a site y if there exist $z_0, \ldots, z_n$ such that $z_0 = x$, $z_n = y$, and there is an open arrow from z_{k-1} to z_k for each $k = 1, \ldots, n$. If this is the case, then we write $x \to y$, where by definition $x \to x$ for all x. Note that $x \to y$ implies $x \leq y$, i.e., all paths walk 'upwards'.

Oriented percolation is a useful model in the study of the dynamics of populations of organisms. To see this, we must interpret the $(1, \ldots, 1)$ direction as our 'time' direction, and all directions perpendicular to this as 'spatial' directions. For simplicity, we consider only the case $d = 2$; in higher dimensions the considerations are similar, but somewhat more complicated to formulate.

If we turn $\mathbb{Z}^2$ anticlockwise over an angle of $\pi/4$ and multiply all lengths by $\sqrt{2}$, we obtain $\mathbb{Z}^2_{\text{even}}$, the even sublattice of $\mathbb{Z}^2$, defined as $\mathbb{Z}^2_{\text{even}} := \{(k,t) \in \mathbb{Z}^2 : k + t \text{ is even}\}$. Viewing our directed graph this way, there are two arrows starting from each point $(k,t) \in \mathbb{Z}^2_{\text{even}}$, one ending at $(k-1, t+1)$ and the other at $(k+1, t+1)$; as before each arrow is open with probability p and closed with the remaining probability. For each $(k,t) \in \mathbb{Z}^2_{\text{even}}$ and $s \geq t$, we set

$$\eta_s^{(k,t)} := \big\{ l \in \mathbb{Z} : (l,s) \in \mathbb{Z}^2_{\text{even}} \text{ and } (k,t) \to (l,s) \big\}. \tag{1.4}$$

We claim that $(\eta_s^{(k,t)})_{s \geq t}$ is a Markov chain, taking values, in turn, in the finite subsets of the even and odd integers. To verify this, we condition on $(\eta_t^{(k,t)}, \eta_{t+1}^{(k,t)}, \ldots, \eta_s^{(k,t)})$, and, given these sets, consider $\eta_{s+1}^{(k,t)}$. Since

$$\begin{aligned} \eta_{s+1}^{(k,t)} = \big\{ l \in \mathbb{Z} : (l, s+1) \in \mathbb{Z}^2_{\text{even}} &\text{ and } \exists m \in \eta_s^{(k,t)} \text{ such that} \\ &\text{there is an open arrow from } (m,s) \text{ to } (l, s+1) \big\}, \end{aligned} \tag{1.5}$$

the set $\eta_{s+1}^{(k,t)}$ is determined only by $\eta_s^{(k,t)}$ and the open arrows between the times s and $s+1$. Since these arrows are independent of $(\eta_t^{(k,t)}, \eta_{t+1}^{(k,t)}, \ldots, \eta_{s-1}^{(k,t)})$, the conditional distribution of $\eta_{s+1}^{(k,t)}$ given $(\eta_t^{(k,t)}, \eta_{t+1}^{(k,t)}, \ldots, \eta_s^{(k,t)})$ is a function of $\eta_s^{(k,t)}$ only, which means $(\eta_s^{(k,t)})_{s \geq t}$ has the Markov property.

We can interpret this Markov chain as follows. Assume that we are studying a plant in a one-dimensional environment, e.g., a river. At a certain time t, a single plant is introduced at the position k. Each summer, each plant of this species produces two new shoots, one upstream, and one downstream, say, and then dies. These shoots survive the next winter with probability p, and if they do, they grow into new plants that can produce new shoots next summer. In this way the population of plants grows, with the additional rule that if two plants try to produce a fresh shoot at the same position, there is only place for one, so one of these shoots immediately perishes. Denoting the set of positions occupied by plants in the s-th summer by $\eta_s^{(k,t)}$, we arrive at the Markov chain described above.

Returning to general, d-dimensional oriented percolation, let us put

$$\vec{C}(x) := \{ y \in \mathbb{Z}^d : x \to y \}, \tag{1.6}$$

and in analogy with the unoriented case, let us define

$$\vec{\theta}(p) := P_p\big[|\vec{C}(0)| = \infty \big]. \tag{1.7}$$

We can interpret $\vec{\theta}(p)$ as the probability that a population of plants, living in a $d-1$ dimensional environment, starting from a single ancestor, survives for all time. Again, $\vec{\theta}$ is trivial for $d = 1$, so we restrict ourselves to $d \geq 2$ only.

The behavior of $\vec{\theta}$ is in broad lines similar to that in the unoriented case. It is known that $\vec{\theta}$ is nondecreasing as a function of p, and that there exists a *critical parameter* $0 < \vec{p}_{\text{c}} < 1$ (depending on the dimension) such that $\vec{\theta}(p) = 0$ for $p < \vec{p}_{\text{c}}$ but $\vec{\theta}(p) > 0$ for $p > \vec{p}_{\text{c}}$. In contrast with the unoriented case, it is rigorously known that $\vec{\theta}$ is continuous on the whole interval $[0,1]$, so in particular $\vec{\theta}(\vec{p}_{\rm c}) = 0$.

There are again precise conjectures about the behavior of $\vec{\theta}$ near the critical point; in particular, it is believed that

$$\vec{\theta}(p) \approx (p - \vec{p}_{\rm c})^{\vec{\beta}} \quad \text{as } p \downarrow \vec{p}_{\rm c}, \tag{1.8}$$

where $\vec{\beta} = \vec{\beta}(d) > 0$ are critical exponents, depending on the dimension. These critical exponents are different from those in the unoriented case. It is believed that $\vec{\beta}(d) = 1$ for all $d \geq 5$; this has been proved rigorously in sufficiently high dimensions (see [HS05]). For dimensions 2 through 4, only numerical estimates for the $\vec{\beta}(d)$'s are known (for example: $\vec{\beta}(2) \cong 0.277$); not even their existence has been rigorously proved.

1.3 Site percolation, other lattices

Instead of letting the bonds between sites be open or closed, we could decide that the sites themselves can be open or closed, with probability p or $1-p$, respectively, independently for all sites. In this case, we say that there is a *path of open sites* from x to y, denoted by $x \leftrightarrow y$, if there exist open sites $z_0, \ldots, z_n$ such that $z_0 = x$, $z_n = y$, and $|z_{k-1} - z_k| = 1$ for each $k = 1, \ldots, n$. Here $x \not\leftrightarrow x$ if x is closed.

The model we obtain in this way is called (unoriented) *site percolation.* Defining the open cluster at x in site percolation as $C^{\rm site}(x) := \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$, we can define a percolation probability $\theta^{\rm site}(p) := P_p[|C^{\rm site}(0)| = \infty]$ as before. By restricting our attention to upward paths only, i.e., paths such that $z_{k-1} \leq z_k$ for each $k = 1, \ldots, n$, we obtain a model that is known as *oriented* (or *directed*) *site percolation.*

It turns out that site percolation behaves in much the same way as bond percolation. There exists a critical value $p_{\rm c}^{\rm site}$ such that $\theta^{\rm site}(p) > 0$ for $p > p_{\rm c}^{\rm site}$ and $\theta^{\rm site}(p) = 0$ for $p < p_{\rm c}^{\rm site}$. The value of $p_{\rm c}^{\rm site}$ is not the same as the critical parameter $p_{\rm c}$ for bond percolation, but the overall shape of $\theta^{\rm site}$ is the same. This is even believed to apply to the critical exponents, i.e., it is believed that

$$\theta^{\rm site}(p) \approx (p - p_{\rm c}^{\rm site})^{\beta} \quad \text{as } p \downarrow p_{\rm c}^{\rm site}, \tag{1.9}$$

where the critical exponent β is *exactly the same* as for bond percolation. Similar statements hold for oriented site percolation.

Although this is so far nonrigorous (except in high dimensions), it demonstrates the principle of *universality* that is quite important in the philosophy of the subject. Roughly speaking, this principle says that up to a certain level, it does not matter precisely how we choose the local laws of our model, since similar local laws will have roughly the same kind of large scale behavior. For example, when we use unoriented bond percolation on $\mathbb{Z}^3$ to model a porous stone, the assumption that cavities in this stone are ordered as in a cubic lattice, with bonds between them that can be open or closed, seems a rather crude simplification of reality. Nevertheless, the belief is that this does not matter too much; we could have chosen site percolation instead of bond percolation, or have ordered our sites according to any other regular 3-dimensional lattice, or have chosen them in some random way with random sizes, and still get roughly the same picture for large-scale effects, such as the occurrence of an infinite connected component.

It is sometimes useful to consider (bond) percolation on other lattices than $\mathbb{Z}^d$ with nearest-neighbor bonds. For this, the theory of percolation on transitive graphs provides a natural framework. For more on this topic, see [LP05]. Recall that an (unoriented) *graph* G is a pair $(\mathcal{V}(G), \mathcal{E}(G))$, where $\mathcal{V}(G)$ is a set, the elements of which are called *vertices*, and $\mathcal{E}(G)$ is a collection of subsets $\{x, y\} \subset \mathcal{V}(G)$ containing exactly two elements, called *edges*. By making edges open or closed with probability p and $1-p$, independently of each other, we can formulate a percolation model on any graph. We will usually only be interested in the case that G is *countably infinite, connected, of bounded degree,* and *transitive*. We now explain what these terms mean.

We say that G is countably infinite if $\mathcal{V}(G)$ is. A graph G is *connected* if there exists no proper subset $D \subset \mathcal{V}(G)$ (i.e., $D \neq \emptyset, \mathcal{V}(G)$) such that no element of D is connected by an edge to an element of $\mathcal{V}(G)\backslash D$. A graph is of *bounded degree* if $\sup_{x\in\mathcal{V}(G)} |\{e \in \mathcal{E}(G) : x \in e\}| < \infty$, i.e., if there is a maximal number of edges going out of a vertex. An *automorphism* of a graph is a bijection $g : \mathcal{V}(G) \to \mathcal{V}(G)$ such that $\big\{\{g(x), g(y)\} : \{x, y\} \in \mathcal{E}(G)\big\} = \mathcal{E}(G)$, i.e., g preserves the set of edges. A graph is called *transitive* if for each $x, y \in \mathcal{V}(G)$, there exists an automorphism g of G such that $g(x) = y$. This means that the graph 'looks the same seen from any vertex'. For example, the set of automorphisms of $\mathbb{Z}^d$ with nearest-neighbor edges is the group generated by the translations and reflections. Since for each $x, y \in \mathbb{Z}^d$, there is a translation that takes x into y, the graph $\mathbb{Z}^d$ is transitive.

We give some more examples of transitive graphs. If in $\mathbb{Z}^2$, we connect not only nearest neighbors by an edge, but also sites x, y such that $(y_1, y_2) = (x_1 + 1, x_2 + 1)$, we obtain the *triangular lattice*. In a similar way, we can add edges between all points at distance $\sqrt{2}$, yielding a 'next nearest neighbor lattice', and so on. There is no need to restrict ourselves to graphs that can be embedded in $\mathbb{Z}^d$. For example, T_d, the infinite tree in which every vertex has precisely $d+1$ neighbors, is an example of a countably infinite, connected, transitive graph of bounded degree.

It is also possible to do oriented percolation on general oriented graphs. Recall that an *oriented graph* G is a pair $(\mathcal{V}(G), \mathcal{A}(G))$, where $\mathcal{V}(G)$ is a set with elements called vertices, and $\mathcal{A}(G)$ is a collection of ordered pairs (x, y) with $x, y \in \mathcal{V}(G)$, called *arrows*. Although it is possible to formulate oriented bond percolation on any oriented graph, this general case has been little studied and is probably difficult. One is usually interested in the case that there is a special 'time' direction (in pictures usually denoted upwards), and all arrows point in this direction.

One systematic way to do this is as follows. Starting with any oriented graph G, we can construct a new oriented graph $G \times \mathbb{Z}$ as follows. We let the vertices of $G \times \mathbb{Z}$ be the set $\mathcal{V}(G \times \mathbb{Z}) := \{(x, t) : x \in \mathcal{V}(G),\ t \in \mathbb{Z}\}$, and we define our set of arrows by $\mathcal{A}(G \times \mathbb{Z}) := \big\{\big((x, t), (y, t+1)\big) : (x, y) \in \mathcal{A}(G),\ t \in \mathbb{Z}\big\}$. A disadvantage of this construction is that $G \times \mathbb{Z}$ needs not be connected, even if G is. In this case, we usually select one connected component.

For example, if G is the oriented graph with vertex set $\mathcal{V}(G) := \mathbb{Z}$ and arrow set $\mathcal{A}(G) := \{(x, x-1), (x, x+1) : x \in \mathbb{Z}\}$, then $G \times \mathbb{Z}$ has two connected components, which are $\mathbb{Z}^2_{\rm even}$ and $\mathbb{Z}^2_{\rm odd}$, equipped with arrows as in Section 1.2.

Exercise Let G be the oriented graph with vertex set $\mathcal{V}(G) := \mathbb{Z}^2$ and arrow set $\mathcal{A}(G) := \big\{\big((x_1, x_2), (x_1-1, x_2)\big), \big((x_1, x_2), (x_1, x_2-1)\big), \big((x_1, x_2), (x_1+1, x_2+1)\big) : (x_1, x_2) \in \mathbb{Z}^2\big\}$. Show that $G \times \mathbb{Z}$ has three connected components, each of which is isomorphic to the oriented graph $\mathbb{Z}^3$ considered in Section 1.2.

2 Basic facts

2.1 Comparison arguments

In this section we start proving some rigorous facts about percolation and oriented percolation on $\mathbb{Z}^d$, with $d \geq 2$. Recall the definitions of $\theta(p)$ and $\vec{\theta}(p)$ in (1.1) and (1.7), respectively.

Lemma 1 (Monotonicity in parameter) *The functions θ and $\vec{\theta}$ and nondecreasing in p.*

Proof Our proof uses a *coupling* between models with different values of p. We will show that it is possible to define percolation models with different values of p on one probability space, in such a way that a bond that is open in the model with the lower value of p is also open in the model with the higher value of p. In fact, we use a trick to construct the models for *all* values of p at the same time.

Recall that $\mathcal{E}(\mathbb{Z}^d)$ is the collection of nearest-neighbor edges in $\mathbb{Z}^d$. Let $(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be independent, uniformly on $[0,1]$ distributed random variables. Say that an edge e is p-open if $U_e \leq p$, and p-closed otherwise. Then, obviously, edges are p-open with probability p, independently of each other. Write $x \stackrel{p}{\leftrightarrow} y$ if there exists a path of p-open edges from x to y, and let $C_p(x) := \{y \in \mathbb{Z}^d : x \stackrel{p}{\leftrightarrow} y\}$ denote the p-open cluster at x. Then for $p \leq p'$, obviously $x \stackrel{p}{\leftrightarrow} y$ implies $x \stackrel{p'}{\leftrightarrow} y$, and therefore $C_p(x) \subset C_{p'}(x)$ a.s. for each x. In particular, $\theta(p) = P[|C_p(0)| = \infty] \leq P[|C_{p'}(0)| = \infty] = \theta(p')$.

The same proof works in the oriented case, with unoriented p-open edges replaced by p-open arrows. ∎

The coupling used in the proof of Lemma 1 has another application.

Lemma 2 (Right continuity) *The functions θ and $\vec{\theta}$ are right continuous.*

Proof With the same conventions as in the proof of the previous lemma, let us write $x \stackrel{p}{\leftrightarrow} \infty$ if $|C_p(x)| = \infty$. Then $\theta(p) = P[0 \stackrel{p}{\leftrightarrow} \infty]$ and

$$\lim_{p'\downarrow p} \theta(p') = \lim_{p'\downarrow p} P[0 \stackrel{p'}{\leftrightarrow} \infty] = P[0 \stackrel{p'}{\leftrightarrow} \infty \ \forall p' > p], \tag{2.1}$$

where we have used the continuity of probability measures with respect to decreasing events. Thus, we need to show that $0 \stackrel{p}{\leftrightarrow} \infty$ if and only if $0 \stackrel{p'}{\leftrightarrow} \infty$ for all $p' > p$. The 'only if' part is trivial, so let us prove the 'if' part.

Let $B_n := \{y \in \mathbb{Z}^d : |y_i| \leq n \ \forall i = 1,\ldots,d\}$ be the block of 'radius' n around the origin, and let $\partial B_n := B_{n+1} \backslash B_n$ denote its 'boundary'. Let us say that $0 \stackrel{p}{\leftrightarrow} \partial B_n$ if $0 \stackrel{p}{\leftrightarrow} y$ for some $y \in \partial B_n$. A little thinking convinces us that $0 \stackrel{p}{\leftrightarrow} \infty$ if and only if $0 \stackrel{p}{\leftrightarrow} \partial B_n$ for all $n \geq 1$.

Now assume that $0 \stackrel{p'}{\leftrightarrow} \infty$ for all $p' > p$. Fix $n \geq 1$. For a path $z = (z_0,\ldots,z_m)$ from 0 to ∂B_n that stays in B_{n+1} and uses each edge only once, let $M(z) := \max\{U_{\{z_k,z_{k-1}\}} : k = 1,\ldots,m\}$ be the maximal value of the U_e's on the edges in z. Since there are only finitely many such paths, there is one path z_0 for which $M(z_0)$ is minimal. By the fact that $0 \overset{p'}{\leftrightarrow} \infty$ for all $p' > p$, it follows that $M(z_0) \leq p$. This proves that $0 \overset{p}{\leftrightarrow} \partial B_n$. Since this is true for each $n \geq 1$, our claim follows.

A similar proof works in the oriented case. ∎

In the next lemma, we compare oriented and unoriented percolation with each other.

Lemma 3 (Comparison of oriented and unoriented percolation) *One has $\theta(p) \geq \vec{\theta}(p)$ for all $p \in [0,1]$.*

Proof Since every oriented path is also an unoriented path, $x \overset{p}{\to} y$ implies $x \overset{p}{\leftrightarrow} y$, and therefore $C_p(x) \subset \vec{C}_p(x)$. ∎

We can also compare percolation probabilities in different dimensions. We write $\theta_d(p)$ and $\vec{\theta}_d(p)$ to denote the percolation probabilities for unoriented and oriented percolation, respectively, in dimension d.

Lemma 4 (Comparison of dimensions) *One has $\theta_d(p) \leq \theta_{d+1}(p)$ and $\vec{\theta}_d(p) \leq \vec{\theta}_{d+1}(p)$ for all $d \geq 1$ and $p \in [0,1]$.*

Proof We can identify $\mathbb{Z}^d$ in a natural way with the set $\{x \in \mathbb{Z}^{d+1} : x_{d+1} = 0\}$. Then, if $x \overset{p}{\leftrightarrow} \infty$ along an open path that stays in $\mathbb{Z}^d$, obviously also $x \overset{p}{\leftrightarrow} \infty$ along an open path in $\mathbb{Z}^{d+1}$. The same argument works for oriented paths. ∎

2.2 Bounds on the critical parameter

We define

$$\begin{aligned} p_{\mathrm{c}} &:= \inf\{p \in [0,1] : \theta(p) > 0\}, \\ \vec{p}_{\mathrm{c}} &:= \inf\{p \in [0,1] : \vec{\theta}(p) > 0\}. \end{aligned} \tag{2.2}$$

Then, in view of Lemma 1, $\theta(p) > 0$ for $p > p_{\mathrm{c}}$ and $\theta(p) = 0$ for $p < p_{\mathrm{c}}$, and likewise, $\vec{\theta}(p) > 0$ for $p > \vec{p}_{\mathrm{c}}$ and $\vec{\theta}(p) = 0$ for $p < \vec{p}_{\mathrm{c}}$. We will show that in dimensions $d \geq 2$, the critical values p_{c} and $\vec{p}_{\mathrm{c}}$ are nontrivial, i.e., $0 < p_{\mathrm{c}} < 1$ and $0 < \vec{p}_{\mathrm{c}} < 1$.

We start by proving that $0 < p_{\mathrm{c}}$ and $0 < \vec{p}_{\mathrm{c}}$. By Lemma 3, $\vec{p}_{\mathrm{c}} \geq p_{\mathrm{c}}$, so it suffices to prove that $p_{\mathrm{c}} > 0$. Recall that $p_{\mathrm{c}}(d)$ denotes p_{c} as a function of the dimension d.

Lemma 5 (Lower bound on critical parameter) *One has*

$$p_{\mathrm{c}}(d) \geq \frac{1}{2d-1} \qquad (d \geq 2). \tag{2.3}$$

Proof If $0 \overset{p}{\leftrightarrow} \infty$, then for each $n \geq 1$ we can find a p-open path of length n, starting at 0 and visiting each site at most once. Let $N_p(n)$ be the number of such paths. Then

$$\theta(p) \leq P[N_p(n) \geq 1] \leq E[N_p(n)] \leq p^n 2d(2d-1)^{n-1} \tag{2.4}$$

for each $n \geq 1$. Here the last inequality follows from the following considerations. A path along n edges, starting at the origin, that visits each site at most once, has in the first step

$2d$, and in each further step at most $2d-1$ edges to choose from; therefore there are at most $2d(2d-1)^{n-1}$ such paths. The probability that all edges in any such path are open is p^n, and therefore the expected number of such open paths is at most $p^n(2d-1)^n$. If $p<(2d-1)^{-1}$, the right-hand side of (2.4) tends to zero as $n\to\infty$, which implies (2.3). ■

Upper bounds on p_c are harder to prove. In view of Lemma 4, in order to show that $p_c(d)<1$ and $\vec{p}_c(d)<1$ for all $d\geq 2$, it suffices to prove these statements in dimension $d=2$. Because of Lemma 3 it even suffices to prove the claim for $\vec{p}_c(2)$ only. Nevertheless, we start with the unoriented case, since the argument is simpler in this case. Our proof is based on counting contours; this type of proof is called a *Peierls argument*, after Rudolf Peierls who used it in 1936 to prove that the Ising model has a phase transition. The next result is our first serious step into percolation theory.

Proposition 6 (Upper bound on critical parameter) *One has* $p_c(2)\leq\frac{2}{3}$.

Proof We need to show that $\theta_2(p)>0$ if $p>\frac{2}{3}$. To this aim, we make use of a form of *duality*. Let $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2}):=\{(n+\frac{1}{2},m+\frac{1}{2}):n,m\in\mathbb{Z}\}$. We connect two points in $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})$ by an edge if they are at distance one. We call $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})$, equipped with a graph structure in this way, the *dual lattice* of $\mathbb{Z}^2$. Note that turning an edge in $\mathbb{Z}^2$ around its middle point over an angle of $\pi/2$ yields an edge in $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})$. We call such edges *dual* to each other. Adopting the convention that an edge in $\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})$ is open if and only if its dual edge is closed, we observe that $0\not\leftrightarrow\infty$ if and only if there is no closed circuit of open dual edges surrounding the origin.

Let $N(n)$ be the number of closed circuits of open dual edges surrounding the origin. Then

$$1-\theta(p)\leq P[N(n)\geq 1\text{ for some }n\geq 4]\leq E\Big[\sum_{n\geq 4}N(n)\Big]\leq\sum_{n\geq 4}n3^n(1-p)^n. \tag{2.5}$$

Here the last inequality follows from the following considerations. An open circuit of n dual edges around the origin, visiting each site at most once, must pass through one of the sites $(\frac{1}{2},\frac{1}{2}),(1+\frac{1}{2},\frac{1}{2}),\ldots,(n-\frac{1}{2},\frac{1}{2})$, and moreover, in each step has 3 edges to choose from where to go next. Therefore, there can be at most $n3^n$ such circuits. Each circuit has probability $(1-p)^n$ to be open, and hence the expected number of open circuits of this length is at most $n3^n(1-p)^n$.

We observe that the right-hand side of (2.5) is finite for $p>\frac{2}{3}$. However, it does not need to be smaller than 1, which is what we need. We therefore adapt our argument a bit. We observe that with probability p^m, all of the edges $\{(0,0),(1,0)\},\ldots,\{(m-1,0),(m,0)\}$ are open. Conditioning on this event, we observe that $0\not\leftrightarrow\infty$ if and only if there is no closed circuit of open dual edges surrounding the sites $\{(0,0),\ldots,(m,0)\}$. Such a circuit must have al least length $2m+2$, so arguing as before we see that

$$\theta(p)\geq p^m\Big(1-\sum_{n\geq 2m+2}n3^n(1-p)^n\Big). \tag{2.6}$$

Since for any $p>\frac{2}{3}$, we can choose m large enough such that the right-hand side of (2.6) is positive, we are done. ■

Proposition 7 (Upper bound on oriented critical parameter) *One has* $\vec{p}_{\rm c}(2) \le \frac{8}{9}$.

Proof As before, we make $\mathbb{Z}^2$ into an oriented graph by connecting sites $x, y \in \mathbb{Z}^2$ at distance $|x - y| = 1$ with an arrow, which points from the lower of the two sites to the higher one. Grabbing these arrows in the middle and turning them anticlockwise over an angle of $\pi/2$ makes the dual lattice $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ into an oriented graph. We let an arrow in $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ be closed if and only if its dual arrow is open.

We now make the following observation. One has $0 \to \infty$ if and only if there is a path along edges in the dual lattice $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, starting at a point $(n - \frac{1}{2}, -\frac{1}{2})$ and ending in a point $(-\frac{1}{2}, m + \frac{1}{2})$ with $n, m \ge 0$, and passing north-east of the origin, such that all edges that are passed in the direction of the arrow are open (while edges that are passed in the other direction may be open or closed).

There are at most $n3^n$ such paths of length n, and these paths pass at least half of their edges in the direction of the arrow. Therefore, the expected number of paths of length n of the type just described is bounded by $n3^n(1-p)^{n/2}$. Observing that $\sum_n n3^n(1-p)^{n/2} < \infty$ if $p > 8/9$, the proof proceeds as in the unoriented case. ∎

We conclude this section by remarking that it is known that $p_{\rm c}(d) \sim (2d)^{-1}$ as $d \to \infty$. This means that in high dimensions, our lower bound on $p_{\rm c}(d)$ is sharp. On the other hand, $p_{\rm c}(2) = \frac{1}{2}$, so our upper bound is not very good. The fact that $p_{\rm c}(2) = \frac{1}{2}$ is a result of duality. To get an idea why this is so, consider the block $A_n := \{0, \ldots, n+1\} \times \{0, \ldots, n\}$ and the dual block $A'_n := \{\frac{1}{2}, \ldots, n + \frac{1}{2}\} \times \{-\frac{1}{2}, \ldots, n + \frac{1}{2}\}$. Let $\partial_{\rm l} A_n := \{0\} \times \{0, \ldots, n\}$ and $\partial_{\rm r} A_n := \{n+1\} \times \{0, \ldots, n\}$ be the left and right boundary of A_n, and let $\partial_- A'_n := \{\frac{1}{2}, \ldots, n + \frac{1}{2}\} \times \{-\frac{1}{2}\}$ and $\partial_+ A'_n := \{\frac{1}{2}, \ldots, n + \frac{1}{2}\} \times \{n + \frac{1}{2}\}$ be the lower and upper boundary of A'_n. Then there is an open path through A_n from a point in $\partial_{\rm l} A_n$ to a point in $\partial_{\rm r} A_n$ if and only if there is no open path through A'_n from a point in $\partial_- A'_n$ to a point in $\partial_+ A'_n$. It follows that

$$P[\partial_{\rm l} A_n \overset{A_n}{\longleftrightarrow} \partial_{\rm r} A_n] + P[\partial_- A'_n \overset{A'_n}{\longleftrightarrow} \partial_+ A'_n] = 1. \tag{2.7}$$

In particular, if $p = \frac{1}{2}$, then by symmetry, both probabilities must be equal to $\frac{1}{2}$. Using such ideas, it is not too hard to prove that $\theta_2(p) > 0$ implies $\theta_2(1-p) = 0$ and hence $p_{\rm c}(2) \ge \frac{1}{2}$. The proof of the opposite inequality is much more involved, however.

The values of $p_{\rm c}(d)$ for $d \ge 3$ are not known explicitly. There exist good rigorous bounds from below, but the best known rigorous upper bounds are not very good.

3 Uniqueness of the infinite cluster

3.1 The number of infinite clusters

In the previous section, we have proved that for p high enough, there is an infinite *connected component* (i.e., open cluster) in unoriented bond percolation on $\mathbb{Z}^d$. We now ask ourselves the question: how many such components are there? Although this sounds unlikely, we have not ruled out the possibility that there exist sites x, y such that $C(x)$ and $C(y)$ are both infinite, but $x \not\leftrightarrow y$. This problem kept probabilists busy for quite long. The solution we present in this section is due to Burton and Keane, [BK89]. This is not the historically first proof (which was given in [AKN87]), but it is a lot easier than earlier proofs, while being more generally applicable.

We start with two simple observations. Our first observation says that the number of connected components is nonrandom.

Lemma 8 (Number of infinite components) *Let N_p be the number of different open clusters in (unoriented) bond percolation on $\mathbb{Z}^d$ with parameter p. There exists a number $n_p = n_p(d) \in \{0, 1, 2, \ldots, \infty\}$, such that $N_p = n_p$ a.s.*

Proof This is a result of the (spatial) ergodicity of product measure on $\mathcal{E}(\mathbb{Z}^d)$. As in the proof of Lemma 1, let $(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be independent, uniformly distributed $[0,1]$-valued random variables, and say that an edge e is p-open if $U_e \le p$. Let $u = (u_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ denote a typical element of $[0,1]^{\mathcal{E}(\mathbb{Z}^d)}$. We equip $[0,1]^{\mathcal{E}(\mathbb{Z}^d)}$ (of course) with the product-σ-field, denoted by $\mathcal{F}$, and define translation operators on $[0,1]^{\mathcal{E}(\mathbb{Z}^d)}$ by

$$(T_x u)_{\{y,z\}} := u_{\{y+x,z+x\}} \qquad (x, y, z \in \mathbb{Z}^d). \tag{3.1}$$

Let $\mathcal{T}$ be the σ-field of translation invariant measurable subsets of $[0,1]^{\mathcal{E}(\mathbb{Z}^d)}$, i.e.,

$$\mathcal{T} := \{A \in \mathcal{F} : T_x^{-1}(A) = A \ \forall x \in \mathbb{Z}^d\}. \tag{3.2}$$

Since the $(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ are independent and identically distributed, their law is (spatially) ergodic, i.e., [1]

$$P[(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)} \in A] \in \{0,1\} \quad \forall A \in \mathcal{T}. \tag{3.3}$$

It follows that any translation invariant measurable function of the $(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ is a.s. constant. This applies in particular to N_p, the number of infinite connected components of the random graph whose vertex set is $\mathbb{Z}^d$ and whose edges are the p-open edges. ∎

Excercise Show that the number N of different open clusters is measurable. Hint: As before, let $B_n := \{-n, \ldots, n\}^d$ and $\partial B_n : B_{n+1}\backslash B_n$. For $n \le m$, let $N(n,m)$ be the number of clusters in B_{m+1} that intersect both ∂B_m and B_n. Now send n and m in the right order to infinity.

Lemma 9 (Zero, one, or infinity) *Let $n_p(d)$ be the number of connected components from Lemma 8. Then $n_p(d) \in \{0, 1, \infty\}$.*

---

[1] This can be proved as follows. A translation invariant probability measure measure μ on $[0,1]^{\mathbb{Z}^d}$ is called *mixing* if for any measurable events $A, B \subset [0,1]^{\mathbb{Z}^d}$ depending on only finitely many coordinates,

$$\lim_{|x|\to\infty} \mu(A \cap T_x^{-1}(B)) = \mu(A)\mu(B),$$

i.e., events that happen far from each other are approximately independent. It is easy to see that product measures are mixing. We claim that mixing measures are ergodic. Here is a sketch of a proof. Let $A \in \mathcal{T}$. Enumerate the coordinates in some way, let $\mathcal{F}_n$ be the σ-field generated by the first n coordinates, and let $f_n := \mu(A|\mathcal{F}_n)$. Then $\lim_{n\to\infty} \int \mu(\mathrm{d}u)|1_A(u) - f_n(u)|^2 = 0$. Using this fact, it is not hard to show that $\mu(A) = \int \mu(\mathrm{d}u)1_A(u)1_A(T_x u) = \int \mu(\mathrm{d}u) f_n(u) f_n(T_x u) + \varepsilon_n = (\int \mu(\mathrm{d}u) f_n(u))^2 + \varepsilon_n + \varepsilon_{n,x} = \mu(A)^2 + \varepsilon_n + \varepsilon_{n,x} + \varepsilon'_n$, where $\varepsilon_n, \varepsilon'_n \to 0$ as $n \to \infty$, and $\varepsilon_{n,x} \to 0$ as $|x| \to \infty$ for fixed n. Letting first $|x| \to \infty$ and then $n \to \infty$ we conclude that $\mu(A) = \mu(A)^2$, hence $\mu(A) \in \{0, 1\}$.

Proof Let $B_m := \{x \in \mathbb{Z}^d : |x_i| \leq m \ \forall i = 1, \ldots, d\}$ denote the block of 'radius' m around the origin. Imagine that $n_p(d) \geq 2$. Then there exists an m and sites $x, y \in B_m$ such that with positive probability, $|C(x)| = \infty$, $|C(y)| = \infty$, and $x \not\leftrightarrow y$. Trivially $n_p(d) = 0$ if $p = 0$, so without loss of generality we may assume that $p > 0$. But then, with positive probability, x and y could be connected in B_m, while outside this box, 'the picture is the same'. Let us try to make this statement more precise.

Let $\mathcal{E}(B_m) := \{\{x, y\} : x, y \in B_m\}$ be the edges between sites in B_m. For any $H \subset \mathcal{E}(B_m)$ and $x, y \in \mathbb{Z}^d$, write $x \leftrightarrow_H y$ if x can be connected to y along a path of edges that either lie outside $\mathcal{E}(B_m)$ and are open, or are in H. Let $C_H(x) := \{y \in \mathbb{Z}^d : x \leftrightarrow_H y\}$ denote the H-cluster at x and let N_H be the number of infinite H-clusters. There exists a subset $H \subset \mathcal{E}_L$ such that the event

$$\big\{|C_H(x)| = \infty,\ |C_H(y)| = \infty,\ x \not\leftrightarrow_H y,\ N_H = n_p,\ \text{and}\ \{e \in \mathcal{E}(B_m) : e \text{ is open }\} = H\big\} \quad (3.4)$$

has positive probability. Let $H' \supset H$ be a subset of $\mathcal{E}_L$ such that x is connected to y along a path of edges in H'. Since $p > 0$, the event

$$\big\{|C_H(x)| = \infty,\ |C_H(y)| = \infty,\ x \not\leftrightarrow_H y,\ N_H = n_p,\ \text{and}\ \{e \in \mathcal{E}(B_m) : e \text{ is open }\} = H'\big\} \quad (3.5)$$

also has positive probability. Let N denote the number of infinite open clusters (not H-clusters). Since in (3.5), we have connected two open clusters that were not connected in (3.4), conditional on the event in (3.5) we have $N \leq n_p - 1$.

Thus, assuming that $n_p \geq 2$, we have shown that

$$P[N \leq n_p - 1] > 0. \quad (3.6)$$

Since $N = n_p$ a.s., this implies $n_p \leq n_p - 1$, which is only possible if $n_p = \infty$. ∎

In view of Lemma 9, in order to show that the infinite cluster, if it exists, is unique, all one has to do is to rule out that there are infinitely many infinite clusters. At this point, probabilists got stuck for almost 15 years. Before we give the solution of Burton and Keane, let us contemplate for a while why the problem is difficult.

So far, our arguments used almost no properties of the lattice $\mathbb{Z}^d$. In fact, Lemmas 8 and 9 remain valid if $\mathbb{Z}^d$ is replaced by any countably infinite, connected, transitive graph. However, the claim that $n_p \in \{0, 1\}$ is not true in this generality!

Lemma 10 (Percolation on a tree) *Let T_d be the infinite tree in which each vertex has $d+1$ neighbors ($d \geq 2$). Then, for $1/d < p < 1$, nearest neighbor bond percolation on T_d has a.s. infinitely many infinite connected components.*

Proof Fix a vertex x, and let η^x_t be the set of vertices y at distance t from x, such that $x \to y$. Then $(|\eta^x_t|)_{t \geq 1}$ is a branching process, where in each generation, each individual produces a $\mathrm{Bin}(d, p)$-distributed number of offspring. If $p > 1/d$, this branching process is supercritical, and hence survives with positive probability. It follows that $x \to \infty$ with positive probability, which proves that the number N of infinite connected components is with positive probability nonzero.

By the same arguments as in Lemmas 8 and 9, either $N = 1$ a.s. or $N = \infty$ a.s. The first possibility is easily ruled out, since it is easy to see that with positive probability there exist vertices x, y such that $x \to \infty$ and $y \to \infty$, but $x \not\leftrightarrow y$. ∎

3.2 Zero or one infinite cluster

Since we have seen in Lemma 10 that on general graphs, it is not always true that $n_p \in \{0,1\}$, we must understand what special property of $\mathbb{Z}^d$ causes the infinite cluster to be unique, and we must find out how to use this property to prove this. It turns out that the important property of $\mathbb{Z}^d$, that trees lack, is that in $\mathbb{Z}^d$, the surface of a large block is small compared to its volume.

Theorem 11 (Uniqueness of the infinite cluster) *Let n_p be as in Lemma 8. Then $n_p \in \{0,1\}$.*

Proof By Lemma 9, it suffices to show that $n_p \neq \infty$. We will assume $n_p = \infty$ and search a contradiction. If $n_p = \infty$, then there exists an m and sites $x, y, z \in B_m$ such that with positive probability, x, y, and z belong to three different infinite clusters. By adding a finite number of open edges in B_m, we can connect these three clusters in such a way that there is a vertex $t \in B_m$ with the property that any open path from one of the three clusters to one of the others must pass through this point. Arguing in the same way as in the proof of Lemma 9, we see that this event must also have positive probability.

What do we learn from this? Call a vertex t a *furcation point* if 1° t is part of an infinite cluster, and 2° removing all open edges touching t splits this infinite cluster in at least three disjoint infinite clusters. By what we have just proved, there exists a $t \in \mathbb{Z}^d$ such that with positive probability, t is a furcation point. By translation invariance,

$$\rho := P[t \text{ is a furcation point}] > 0 \tag{3.7}$$

does not depend on $t \in \mathbb{Z}^d$.

Let $\partial B_m := B_{m+1} \backslash B_m$ denote the 'boundary' of B_m. We claim that the number of furcation points in B_m cannot be larger than $|\partial B_m| - 2$. To see this, close open edges between sites in B_{m+1} as long as this is possible without removing any of the furcation points in B_m. We claim that when we are done, the remaining open edges in B_{m+1} form a collection of disjoint trees (i.e., a forest). Indeed, if there is an open circuit in B_{m+1}, then closing any edge in this circuit will not stop any of the furcation points in B_m from being a furcation point. If there is a vertex in B_m with only one open edge going out, then this edge can also be closed, so what we are left with is a collection of trees whose ends all reach ∂B_m, where each furcation point is a 'branch point' with at least three ends goint out. It is easy to see that the number of ends reaching ∂B_m must at least be equal to the number of furcation points plus two.

By (3.7), the expected number of furcation points in B_m is $\rho|B_m|$, while by what we have just proved, the number of furcation points in B_m never exceeds $|\partial B_m|$. Since $|\partial B_m|/|B_m| \to 0$ as $m \to \infty$, we can choose m sufficiently large such that $\rho|B_m| > |\partial B_m|$, which yields the desired contradiction. ∎

In the proof of Theorem 11, we used in an essential way that the surface $|\partial B_m|$ of a large block is much smaller than its volume $|B_m|$. Let G be a general, countably infinite, connected, transitive graph of bounded degree. Define the surface of a set of vertices $A \subset \mathcal{V}(G)$ as

$$\partial A := \{y \in \mathcal{V}(G) \backslash A : \{x, y\} \in \mathcal{E}(G) \text{ for some } x \in A\}. \tag{3.8}$$

By definition, G is called *amenable* if there exists a sequence of finite, nonempty sets $A_m \subset \mathcal{V}(G)$ such that $|\partial A_m|/|A_m| \to 0$. The proof of Theorem 11 can be generalized to any amenable, countably infinite, connected, transitive graph of bounded degree. Regular trees are an example of nonamenable graphs, and, as we have seen in Lemma 10, uniqueness of the infinite cluster does not hold on trees. Indeed, it has been conjectured that for any nonamenable, countably infinite, connected, transitive graph of bounded degree, there exists values of the percolation parameter p for which there are infinitely many infinite clusters. There are some partial results on this, but the general problem is still open.

Remarkably, in the proof of Theorem 11, we have not used the independence of bonds! All we have used is that the configuration of open and closed bonds is spatially ergodic (in Lemma 8), and has the property that if a certain situation has positive probability, then the situation that arises when we change a finite number of bonds from closed to being open, or vice versa, also has positive probability (we used this in the proof of Lemma 9 and again in the proof of Theorem 11). This latter property is called *finite energy*.

Theorem 11 applies only to undirected percolation. Here is an open problem: formulate an analogue of uniqueness of the infinite cluster for oriented percolation, and prove it for a reasonably large class of graphs. As some step in this direction, Grimmett and Hiemer [GH02] have proved for oriented percolation on $\mathbb{Z}^d$, that if $x \to \infty$ and $y \to \infty$, then a.s. there exists a point z such that $x \to z$, $y \to z$, and $z \to \infty$.

3.3 Continuity of the percolation probability

As an application of Theorem 11, we prove that the percolation probability θ is continuous on $(p_{\rm c}, 1]$.

Proposition 12 (Continuity of the percolation probability) *The percolation probability θ is continuous on $(p_{\rm c}, 1]$.*

Proof Right continuity has been proved in Lemma 2, so it suffices to prove left continuity. As before, let $(U_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ be independent, uniformly distributed $[0,1]$-valued random variables, but changing our earlier definition a bit, say that an edge e is p-open if $U_e < p$ (instead of $U_e \leq p$). Then, obviously, it is still true that edges are independently of each other p-open with probability p. Now

$$\lim_{p'\uparrow p}\theta(p') = \lim_{p'\uparrow p} P[0 \overset{p'}{\leftrightarrow} \infty] = P[\exists p' < p \ \text{ s.t. } \ 0 \overset{p'}{\leftrightarrow} \infty]. \tag{3.9}$$

Assuming that $p > p_{\rm c}$, we need to show that $0 \overset{p}{\leftrightarrow} \infty$ if and only if $0 \overset{p'}{\leftrightarrow} \infty$ for some $p' < p$. The 'if' part is trivial, so let us prove the 'only if' part.

Assume that $0 \overset{p}{\leftrightarrow} \infty$. Since $p > p_{\rm c}$, there exists a $p'' < p$ such that for some $x \in \mathbb{Z}^d$, the p''-cluster $C_{p''}(x)$ is infinite. Since by Theorem 11, the infinite p-cluster is unique, we must have $C_{p''}(x) \subset C_p(0)$. It follows that there exists a p-open path $z = (z_0, \ldots, z_m)$ from 0 to x. Let $M(z) := \max\{U_{\{z_k,z_{k-1}\}} : k = 1, \ldots, k\}$. Since the path is p-open (according to our new definition), $M(z) < p$. But then there exists a $p'' < p' < p$ such that the path is also p'-open, and hence $0 \overset{p'}{\leftrightarrow} \infty$. ∎

A disadvantage of the proof of Lemma 12 is that it works only for graphs for which uniqueness of the infinite cluster holds. It also does not work for oriented percolation. We will see how to prove the analogue of Lemma 12 for oriented percolation later on. In order to prove that θ is continuous at $p_{\rm c}$, it suffices to prove that there is no infinite cluster at $p_{\rm c}$, but so far nobody knows how to do this in dimensions 3–6.[2]

4 Oriented percolation

4.1 The upper invariant measure

In this section, we are going to take a closer look at oriented percolation. We will mainly be interested in the long-time behavior of the Markov chain $(\eta_t)_{t\geq 0}$ from (1.4), and its analogues in higher dimension (see the exercise at the end of Section 1.3). In studying the Markov chain from (1.4), it is somewhat inconvenient that η_t takes values in different spaces at even and odd times. We therefore change our model a bit. All our arguments will also work for the model from Section 1.2, but they are a bit more complicated to formulate there.

In this section, we look at the following model. We write $\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z}$ and denote a point in $\mathbb{Z}^{d-1} \times \mathbb{Z}$ by (x,t), where $x \in \mathbb{Z}^{d-1}$ is our spatial coordinate, and t denotes time. We make $\mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z}$ into an oriented graph by drawing an arrow from a point (x,t) to a point (y,s) if $|x-y| \leq 1$ and $s = t+1$. As before, we let arrows independently of each other be open with probability p and closed with the remaining probability, and we write $(x,t) \to (y,s)$ if there exists an oriented path of open arrows from (x,t) to (y,s). In our present model, we define our oriented percolation probability as

$$\vec{\theta}'_d(p) := P_p[0 \to \infty] \tag{4.1}$$

and our critical percolation parameter as

$$\vec{p}'_{\rm c}(d) := \inf\{p \in [0,1] : \vec{\theta}'_d(p) > 0\}, \tag{4.2}$$

where we use an accent to distinguish these quantities from those concerning the model from Section 1.2. Simple comparison arguments show that $\vec{\theta}'_d(p) \geq \vec{\theta}'_d(2) \geq \vec{\theta}_d(2)$, so $p'_{\rm c}(d) < 1$. On the other hand, adapting the proof of Lemma 5 a bit, we find that $p'_{\rm c}(d) \geq (2d-1)^{-1} > 0$.

We will use the following notation: if $C, D \subset \mathbb{Z}^{d-1} \times \mathbb{Z}$, then we write $C \to D$ if there exist $c \in C$ and $d \in D$ such that $c \to d$. We write $c \to D$ and $C \to d$ as a shorthand for $\{c\} \to D$ and $C \to \{d\}$. For any $C \subset \mathbb{Z}^{d-1} \times \mathbb{Z}$ and $t \in \mathbb{Z}$, we define

$$\eta^C_t := \{y \in \mathbb{Z}^{d-1} : C \to (y,t)\}. \tag{4.3}$$

Then, for any $A \subset \mathbb{Z}^{d-1}$ and $s \in \mathbb{Z}$, the process

$$\big(\eta^{A\times\{s\}}_t\big)_{t\geq s} \tag{4.4}$$

[2] It is known, however, that at $p_{\rm c}$ there are no paths $0 \leftrightarrow \infty$ that stay entirely in the upper half space $\{x \in \mathbb{Z}^d : x_1 \geq 0\}$.

is a (time-homogeneous) Markov chain, taking values in the subsets of $\mathbb{Z}^{d-1}$, and started at time s in $\eta_s^{A\times\{s\}} = A$.

We remind the reader of the definition of a (time-homogeneous) *Markov chain* $(X_t)_{t\geq 0}$ with *state space* E and *transition probability* $P_t(x, \mathrm{d}y)$. Let $(E, \mathcal{F})$ be a measurable space. By definition, a *probability kernel* $K(x, \mathrm{d}y)$ on E is a function $K : E \times \mathcal{F} \to \mathbb{R}$ such that for fixed $x \in E$, $K(x, \cdot\,)$ is a probability measure on E, and for fixed $A \in \mathcal{F}$, $K(\,\cdot\,, A)$ is a measurable function on E. A *transition probability* on E is a family of probability kernels $P_t(x, \mathrm{d}y)$ $(t = 0, 1, \ldots)$ on E such that $P_0(x, \cdot\,) = \delta_x$ (the delta-measure in the point x) and

$$\int P_s(x, \mathrm{d}y) P_t(y, \mathrm{d}z) = P_{s+t}(x, \mathrm{d}z). \tag{4.5}$$

Let $P_t(x, \mathrm{d}y)$ be a transition probability on E. Then, for each probability measure μ on E, there exists a unique (in distribution) E-valued process $X = (X_t)_{t\geq 0}$ such that

$$P[X_0 \in A_0, \ldots, X_t \in A_t] = \int_{A_0} \mu(\mathrm{d}x_0) \int_{A_1} P_1(x_0, \mathrm{d}x_1) \cdots \int_{A_t} P_1(x_{t-1}, \mathrm{d}x_t) \tag{4.6}$$

for all $A_0, \ldots, A_t \in \mathcal{F}$ and $t \geq 0$. We call X the *Markov chain* in E with *transition probability* $P_t(x, \mathrm{d}y)$ and *initial law* μ. We call X_0 the *initial state* of our Markov chain.

We will be interested in the case that E is the space of all subsets of $\mathbb{Z}^{d-1}$. We note that this space can in a natural way be associated with the space $\{0,1\}^{\mathbb{Z}^{d-1}}$, where we identify a set $A \subset \mathbb{Z}^{d-1}$ with its indicator function $1_A(i) := 1$ if $i \in A$ and $1_A(i) := 0$ otherwise. We equip $\{0,1\}^{\mathbb{Z}^{d-1}}$ (of course) with the product-σ-field. We observe that

$$P_t(A, \cdot\,) := P\big[\eta_{s+t}^{A\times\{s\}} \in \cdot\,\big] \tag{4.7}$$

defines a transition probability on $\{0,1\}^{\mathbb{Z}^{d-1}}$. We call a Markov chain in $\{0,1\}^{\mathbb{Z}^{d-1}}$ with this transition probability, and arbitrary initial law, an *oriented percolation process*, and use the symbol $\eta = (\eta_t)_{t\geq 0}$ to denote it. Thus, η may be any Markov chain with the transition probability from (4.7). In particular, $(\eta_t^{A\times\{0\}})_{t\geq 0}$ is an oriented percolation process with deterministic initial state A, which is constructed in a special way, with the help of the oriented percolation model described above.

We can interpret the oriented percolation process as a model for the spread of an organism living in $d-1$ dimensions. Sites in η_t are at time t occupied by an organism at time t; all other sites are vacant. In each time step, an occupied site tries to invade all sites at distance less or equal than 1 from itself (including itself). Such an invasion is succesfull with probability p. If at least one site succesfully invades a given site, the latter is occupied at the next time unit; otherwise it is vacant.

By definition, an *invariant law* for the oriented percolation process is a probability law ν on the space of all subsets of $\mathbb{Z}^{d-1}$, such that $P[\eta_0 \in \cdot\,] = \nu$ implies $P[\eta_t \in \cdot\,] = \nu$ for all $t \geq 0$. This is equivalent to $\nu(\mathrm{d}B) = \int \nu(\mathrm{d}A) P_t(A, \mathrm{d}B)$ $(t \geq 0)$ where $P_t(A, \mathrm{d}B)$ is the transition probability in (4.7), i.e., ν is an invariant law if and only if

$$\nu = \int \nu(\mathrm{d}A) P\big[\eta_{s+t}^{A\times\{s\}} \in \cdot\,\big]. \tag{4.8}$$

Important questions are: how many invariant measures of the oriented percolation process do there exist? What happens if we start the oriented percolation process in some other initial law, and let time tend to infinity?

We start with some simple observations. Obviously the delta measure $\delta_\emptyset$ on the empty set is an invariant law for the oriented percolation process. The next lemma describes an invariant law that is potentially different from $\delta_\emptyset$. Below, we use the notation $-\infty \to (y,t)$ to denote the event that the set $\{(x,s) \in \mathbb{Z}^d : (x,s) \to (y,t)\}$ is infinite.

Lemma 13 (Upper invariant law) *Set* $\eta_t^{\{-\infty\}} := \{y \in \mathbb{Z}^{d-1} : -\infty \to (y,t)\}$ $(t \in \mathbb{Z})$. *Then* $\overline{\nu} := P[\eta_t^{\{-\infty\}} \in \cdot\,]$ *is an invariant law for the oriented percolation process. Moreover,*

$$P[\eta_t^{\mathbb{Z}^{d-1}\times\{0\}} \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{4.9}$$

and if ν *is another invariant law for the oriented percolation process, then* $\nu \leq \overline{\nu}$, *where* $\leq$ *denotes the stochastic order of probability measures.*

Before give the proof of Lemma 13, we first need to explain what the notation $\Rightarrow$ in (4.9) means, and what we mean by 'stochastic order'.

As before, we identify the space of all subsets of $\mathbb{Z}^{d-1}$ with $\{0,1\}^{\mathbb{Z}^{d-1}}$. We equip $\{0,1\}^{\mathbb{Z}^{d-1}}$ with the product topology. We say that a sequence μ_n of probability measures on $\{0,1\}^{\mathbb{Z}^{d-1}}$ *converges weakly* to a limit μ, denoted $\mu_n \Rightarrow \mu$, if

$$\int \mu_n(\mathrm{d}A) f(A) \underset{n\to\infty}{\longrightarrow} \int \mu(\mathrm{d}A) f(A) \tag{4.10}$$

for all continuous $f : \{0,1\}^{\mathbb{Z}^{d-1}} \to \mathbb{R}$. Note that by Tychonoff's theorem, $\{0,1\}^{\mathbb{Z}^{d-1}}$ is compact, so every continuous real function on $\{0,1\}^{\mathbb{Z}^{d-1}}$ is bounded. It follows from general theory that the space of all probability measures on $\{0,1\}^{\mathbb{Z}^{d-1}}$, equipped with the topology of weak convergence, is compact. Using this, it is not hard to see that in order to check that $\mu_n \Rightarrow \mu$, it suffices to check (4.10) for functions depending on finitely many coordinates.

A function $f : \{0,1\}^{\mathbb{Z}^{d-1}} \to \mathbb{R}$ is *nondecreasing* if $A \leq B$ implies $f(A) \leq f(B)$. (In set notation, this says that $A \subset B$ implies $f(A) \leq f(B)$.) It turns out that the following three conditions on probability measures μ_1, μ_2 on $\{0,1\}^{\mathbb{Z}^{d-1}}$ are equivalent:

(i) $\int \mu_1(\mathrm{d}A) f(A) \leq \int \mu_2(\mathrm{d}A) f(A)$ for all continuous nondecreasing f,

(ii) $\int \mu_1(\mathrm{d}A) f(A) \leq \int \mu_2(\mathrm{d}A) f(A)$ for all bounded measurable nondecreasing f,

(iii) There exists a probability measure μ on $\{0,1\}^{\mathbb{Z}^{d-1}} \times \{0,1\}^{\mathbb{Z}^{d-1}}$ with marginals ν_1, ν_2 that is concentrated on $\{(A,B) : A \leq B\}$.

(4.11)

Recall that the *marginals* of a measure on a product space are its projections on the single spaces. Thus, condition (iii) can be reformulated as

(iii)′ Two random variables ξ_1, ξ_2 with laws ν_1, ν_2 can be coupled such that $\xi_1 \leq \xi_2$.

(4.12)

The implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) in (4.11) are easy, but the step (i)$\Rightarrow$(iii) is a bit harder. Since we will only need to use the formulation (iii) of stochastic order, we will actually not need the equivalence of (i)–(iii). For a proof of this, we refer the reader to [Lig85, Theorem II.2.4].

Proof of Lemma 13 Note that $\eta_t^{\{-\infty\}}$ is independent of the arrows between time t and time $t+1$. Therefore, and since $\eta_{t+1}^{\{-\infty\}} = \{y \in \mathbb{Z}^{d-1} : \eta_t^{\{-\infty\}} \to (y, t+1)\}$, we see that $(\eta_t^{\{-\infty\}})_{t\in\mathbb{Z}}$ is a stationary Markov chain with the transition probability from (4.7). In other words, $(\eta_t^{\{-\infty\}})_{t\in\mathbb{Z}}$ is a stationary oriented percolation process, and its law $\overline{\nu} := P[\eta_t \in \cdot\,]$ is an invariant law.

To prove (4.9), we observe that $\eta_t^{\mathbb{Z}^{d-1}\times\{0\}}$ has the same distribution as $\eta_0^{\mathbb{Z}^{d-1}\times\{-t\}}$, and that

$$\eta_0^{\mathbb{Z}^{d-1}\times\{-t\}} \downarrow \eta_0^{\{-\infty\}} \quad \text{as } t \uparrow \infty \quad \text{a.s.,} \tag{4.13}$$

which implies $P[\eta_0^{\mathbb{Z}^{d-1}\times\{-t\}} \in \cdot\,] \Rightarrow P[\eta_0^{\{-\infty\}} \in \cdot\,]$ as $t \to \infty$. Now imagine that ν is another invariant law of the oriented percolation process. Then

$$\nu = \int \nu(\mathrm{d}A) P\big[\eta_0^{A\times\{-t\}} \in \cdot\,\big] \leq P\big[\eta_0^{\mathbb{Z}^{d-1}\times\{-t\}} \in \cdot\,\big], \tag{4.14}$$

where $\leq$ denotes stochastic order. Letting $t \uparrow \infty$, using (4.9) and observing that weak limits preserve the stochastic order, we find that $\nu \leq \overline{\nu}$. ∎

Lemma 13 has a simple corollary.

Corollary 14 (Nonzero invariant laws) *The law $\delta_\emptyset$ is the unique invariant law of the oriented percolation process if and only if $\vec{\theta}'(p) = 0$.*

Proof Since $P[i \in \eta_t^{\{-\infty\}}] = P[-\infty \to (i,t)] = P[(i,t) \to \infty] = \vec{\theta}'(p)$, we see that $\overline{\nu} = \delta_\emptyset$ if and only if $\vec{\theta}'(p) = 0$. Therefore, obviously, if $\vec{\theta}'(p) > 0$ there exists at least one invariant law of the oriented percolation process that is not $\delta_\emptyset$, namely $\overline{\nu}$. On the other hand, if $\vec{\theta}'(p) = 0$ and therefore $\overline{\nu} = \delta_\emptyset$, then, since $\overline{\nu}$ is the largest invariant law in the stochastic order, any other invariant law ν must satisfy $\delta_\emptyset \leq \nu \leq \delta_\emptyset$, which implies $\nu = \delta_\emptyset$. ∎

4.2 Convergence to the upper invariant measure

For any finite $A \subset \mathbb{Z}^{d-1}$, let us write

$$\rho(A) := P[A \times \{0\} \to \infty] = P[\eta_t^{A\times\{0\}} \neq \emptyset \ \forall t \geq 0] \tag{4.15}$$

for the probability that the oriented percolation process started in A survives. Note that $\rho(A) > 0$ for all nonempty A if $\vec{\theta}'(p) > 0$. It is known that

$$P[\eta_t^{A\times\{0\}} \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \rho(A)\overline{\nu} + (1-\rho(A))\delta_\emptyset \tag{4.16}$$

for every finite $A \subset \mathbb{Z}^{d-1}$. Formula (4.16) says that the law of $\eta_t^{A\times\{0\}}$ converges, as $t \to \infty$, to a mixture of the upper invariant law and the empty configuration. In particular, since

$P[\eta_t^{A\times\{0\}} = \emptyset]$ increases as $t \uparrow \infty$ to $(1-\rho(A))$, if the oriented percolation process survives with positive probability, then formula (4.16) is equivalent to the statement that

$$P[\eta_t^{A\times\{0\}} \in \cdot \,|\, \eta_t^{A\times\{0\}} \neq \emptyset] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}, \tag{4.17}$$

i.e., the law of the process $\eta_t^{A\times\{0\}}$ conditioned on survival converges to the upper invariant law. Roughly speaking, this means that if the process survives, then we end up with organisms occupying all parts of space at the highest possible density. Formula (4.16) is known as *complete convergence*.

Complete convergence is hard to prove, so in these notes we will settle for something easier. We will start the oriented percolation process in a translation invariant initial law, that already has occupied sites everywhere, although possibly in a very low density, and show that the law of this process converges to the upper invariant law. Our proof follows the analoguous proof for the contact process due to Harris [Har76, Theorem 9.2] (see also [Lig85, Theorem VI.4.8]).

To prepare for this, we define translation operators on the space of all subsets of $\mathbb{Z}^{d-1}$ by

$$T_i(A) := \{j+i : j \in A\} \qquad (i \in \mathbb{Z}^{d-1},\ A \subset \mathbb{Z}^{d-1}). \tag{4.18}$$

We call a probability measure μ on the space of all subsets of $\mathbb{Z}^{d-1}$ *translation invariant* or (spatially) *homogeneous* if $\mu = \mu \circ T_i^{-1}$ for all $i \in \mathbb{Z}^{d-1}$. We will say that μ is *nontrivial* if $\mu(\{\emptyset\}) = 0$.

Theorem 15 (Convergence from homogeneous initial laws) *Let η be an oriented percolation process started in an nontrivial homogeneous initial law $P[\eta_0 \in \cdot\,] = \mu$. Then*

$$P[\eta_t \in \cdot\,] \underset{t\to\infty}{\Longrightarrow} \overline{\nu}. \tag{4.19}$$

We will split the proof of Theorem 15 into a few lemmas. Our first lemma gives a useful characterization of $\overline{\nu}$.

Lemma 16 (Characterization of the upper invariant law) *The upper invariant law $\overline{\nu}$ is the unique probability measure on the space of all subsets of $\mathbb{Z}^{d-1}$ such that*

$$\int \overline{\nu}(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} = \rho(B) \tag{4.20}$$

for all finite $B \subset \mathbb{Z}^{d-1}$.

Proof By the definition of $\overline{\nu}$

$$\int \overline{\nu}(\mathrm{d}A) 1_{\{A\cap B\neq\emptyset\}} = P[\eta_0^{\{-\infty\}} \cap B \neq \emptyset] = P[-\infty \to B \times \{0\}] = \rho(B). \tag{4.21}$$

This proves (4.20). To see that $\overline{\nu}$ is uniquely determined by this formula, we consider the class $\mathcal{G}$ of all functions of the form

$$f(A) = \sum_{k=1}^{n} a_k 1_{\{A\cap B_k=\emptyset\}}, \tag{4.22}$$

where $B_1, \ldots, B_k$ are finite subsets of $\mathbb{Z}^{d-1}$. Obviosuly (4.20) implies $\int \overline{\nu}(\mathrm{d}A) 1_{\{A\cap B=\emptyset\}} = 1 - \rho(B)$, so by this formula we know the value of $\int \overline{\nu}(\mathrm{d}A) f(A)$ for all $f \in \mathcal{G}$. Since $1_{\{A\cap B=\emptyset\}} 1_{\{A\cap B'=\emptyset\}} = 1_{\{A\cap(B\cup B')=\emptyset\}}$, the class $\mathcal{G}$ is an algebra. Since for each $A \neq A'$ we can find a B such that $1_{\{A\cap B=\emptyset\}} \neq 1_{\{A'\cap B=\emptyset\}}$, the class $\mathcal{G}$ separates points. Therefore, by the Stone-Weierstrass theorem, $\mathcal{G}$ is dense in the space of all continuous functions on the space of all subsets of $\mathbb{Z}^{d-1}$ (which we identify with $\{0,1\}^{\mathbb{Z}^{d-1}}$ as before), and therefore a probability measure on $\{0,1\}^{\mathbb{Z}^{d-1}}$ is uniquely determined by the expectation of functions in $\mathcal{G}$. ∎

We now state two more lemmas, show how they can be used to prove Theorem 15, and then prove the lemmas themselves.

Lemma 17 (Extinction versus unbounded growth) *An oriented percolation process η started in a finite initial state η_0 satisfies*

$$P\big[\exists t \geq 0 \ s.t. \ \eta_t = \emptyset \quad or \quad \lim_{t\to\infty} |\eta_t| = \infty\big] = 1. \tag{4.23}$$

Lemma 18 (Occupied sites everywhere) *Let η be an oriented percolation process started in an nontrivial homogeneous initial law $P[\eta_0 \in \cdot\,] = \mu$. Then*

$$\lim_{t\to\infty} \limsup_{N\to\infty} \sup_{|B|\geq N} P\big[\eta_t \cap B = \emptyset\big] = 0. \tag{4.24}$$

Proof of Theorem 15 Since the space of all probability measures on $\{0,1\}^{\mathbb{Z}^{d-1}}$ is compact, we can select a subsequence of times $t_n \to \infty$ such that along this sequence, the laws $P[\eta_{t_n} \in \cdot\,]$ converge to some limit law. If we can show that this limit law is $\overline{\nu}$ for any such subsequence, we will have proved (4.19). Therefore, by Lemma 16, it suffices to show that

$$\lim_{t\to\infty} P\big[\eta_t \cap B \neq \emptyset\big] = \rho(B) \tag{4.25}$$

for all finite $B \subset \mathbb{Z}^{d-1}$. For any $C \subset \mathbb{Z}^{d-1} \times \mathbb{Z}$ and $t \in \mathbb{Z}$, let us write

$$\xi_t^C := \{y \in \mathbb{Z}^{d-1} : (y,t) \to C\}. \tag{4.26}$$

Then, for any $0 \leq s \leq t$

$$\begin{aligned} P\big[\eta_t \cap B \neq \emptyset\big] &= \int \mu(\mathrm{d}A) P\big[\eta_t^{A\times\{0\}} \cap B \neq \emptyset\big] \\ &= \int \mu(\mathrm{d}A) P\big[A \times \{0\} \to B \times \{t\}\big] \\ &= \int \mu(\mathrm{d}A) P\big[\eta_s^{A\times\{0\}} \cap \xi_s^{B\times\{t\}} \neq \emptyset\big]. \end{aligned} \tag{4.27}$$

Since $\eta_s^{A\times\{0\}}$ and $\xi_s^{B\times\{t\}}$ are defined using arrows in disjoint parts of space, they are independent. Therefore, since ξ is an oriented percolation process 'run backwards in time', we have

$$P\big[\eta_t \cap B \neq \emptyset\big] = P\big[\eta_s \cap \tilde{\eta}_{t-s}^B \neq \emptyset\big] \tag{4.28}$$

where $\tilde{\eta}^B$ is an oriented percolation process started in $\tilde{\eta}^B_0 = B$, independent of η. Now

$$\begin{aligned}P\big[\eta_t \cap B \neq \emptyset\big] = {} & P\big[\eta_s \cap \tilde{\eta}^B_{t-s} \neq \emptyset \,\big|\, \tilde{\eta}^B_{t-s} = \emptyset\big]P\big[\tilde{\eta}^B_{t-s} = \emptyset\big] \\ & + P\big[\eta_s \cap \tilde{\eta}^B_{t-s} \neq \emptyset \,\big|\, 0 < |\tilde{\eta}^B_{t-s}| < N\big]P\big[0 < |\tilde{\eta}^B_{t-s}| < N\big] \\ & + P\big[\eta_s \cap \tilde{\eta}^B_{t-s} \neq \emptyset \,\big|\, |\tilde{\eta}^B_{t-s}| \geq N\big]P\big[|\tilde{\eta}^B_{t-s}| \geq N\big].\end{aligned} \tag{4.29}$$

Obviously, the first term on the right-hand side is zero, while by Lemma 17,

$$\begin{aligned}&\lim_{t\to\infty} P\big[0 < |\tilde{\eta}^B_{t-s}| < N\big] = 0, \\ &\lim_{t\to\infty} P\big[|\tilde{\eta}^B_{t-s}| \geq N\big] = \rho(B).\end{aligned} \tag{4.30}$$

By Lemma 18, for each $\varepsilon > 0$ we can choose $s \geq 0$ and $N \geq 0$ such that

$$\inf_{|B|\geq N} P\big[\eta_s \cap B = \emptyset\big] \leq \varepsilon, \tag{4.31}$$

and therefore

$$P\big[\eta_s \cap \tilde{\eta}^B_{t-s} \neq \emptyset \,\big|\, |\tilde{\eta}^B_{t-s}| \geq N\big] \leq \varepsilon. \tag{4.32}$$

Inserting (4.30) and (4.32) into (4.29) we see that

$$\limsup_{t\to\infty} \big|P\big[\eta_t \cap B \neq \emptyset\big] - \rho(B)\big| \leq \varepsilon. \tag{4.33}$$

Since ε is arbitrary, (4.25) follows. ∎

Proof of Lemma 17 If $p = 1$ the statement is trivial, so let us assume $p < 1$. We have to show that for each fixed $N \geq 1$, the event $1 \leq |\eta_t| \leq N$ happens at only finitely many times t. If $|A| \leq N$, then, since it may happen that all arrows going out of $\eta^{A\times\{0\}}_t$ are closed,

$$P[\eta^{A\times\{0\}}_1 = \emptyset] \geq (1-p)^{(2d+1)N} =: \varepsilon > 0. \tag{4.34}$$

Therefore, if the event $1 \leq |\eta_t| \leq N$ happens for the k-th time, the probability that $\eta_{t+1} = \emptyset$, and therefore $1 \leq |\eta_t| \leq N$ will not happen any more, is at least ε. Thus,

$$P\big[1 \leq |\eta_t| \leq N \text{ at more than } k \text{ times } t\big] \leq \varepsilon^k. \tag{4.35}$$

Letting $k \uparrow \infty$, the claim follows. ∎

Proof of Lemma 18 The naive heuristic idea behind Lemma 18 is that since the law of η_t is nontrivial and homogeneous, it has particles 'everywhere', so if B is sufficiently large, the chance that it has a nonempty intersection with η_t must be close to one. This intuition is, however, a bit too simple, as is demonstrated by the following counterexample: let $d = 2$ and $\eta_0 = \{k \in \mathbb{Z} : k \text{ even }\}$ with probability 1/2 and $\eta_0 = \{k \in \mathbb{Z} : k \text{ odd }\}$, also with probability 1/2. Then the law of η_0 is nontrivial and homogeneous, but setting $B := \{2k : k = 1, \ldots, n\}$, we see that $P[\eta_0 \cap B = \emptyset] = 1/2$, now matter how large n is. Because of this difficulty, we need to run the process η for some time t, in order to sufficiently 'smear out' the occupied sites.

Thus, we wish to show that for every $\varepsilon > 0$ we can choose T large enough, such that for N large enough, $|B| \geq N$ implies

$$\int \mu(\mathrm{d}A) P\big[\eta_T^{A\times\{0\}} \cap B = \emptyset\big] \leq \varepsilon. \tag{4.36}$$

We start by observing that for any finite $B \subset \mathbb{Z}^{d-1}$ and $L \geq 1$, we can choose $\tilde{B} \subset B$ such that $|x-y| > 2L$ for all $x, y \in \tilde{B}$, $x \neq y$, and $|\tilde{B}| \geq |B|(2L+1)^{-(d-1)}$. Choosing such a subset with $L = T$, we have, for any set B with $|B| \geq N$,

$$\begin{aligned}
\int \mu(\mathrm{d}A) P\big[\eta_T^{A\times\{0\}} \cap B = \emptyset\big] &= \int \mu(\mathrm{d}A) P\big[A \times \{0\} \not\to B \times \{T\}\big] \\
&\leq \int \mu(\mathrm{d}A) P\big[A \times \{0\} \not\to (i,T)\ \forall i \in \tilde{B}\big] \\
&= \int \mu(\mathrm{d}A) \prod_{i\in\tilde{B}} P\big[A \times \{0\} \not\to (i,T)\big] \\
&\leq \Big(\prod_{i\in\tilde{B}} \int \mu(\mathrm{d}A) P\big[A \times \{0\} \not\to (i,T)\big]^{|\tilde{B}|}\Big)^{1/|\tilde{B}|} \\
&= \int \mu(\mathrm{d}A) P\big[A \times \{0\} \not\to (0,T)\big]^{|\tilde{B}|} \\
&\leq \int \mu(\mathrm{d}A) P\big[A \times \{0\} \not\to (0,T)\big]^{N(2T+1)^{-(d-1)}},
\end{aligned} \tag{4.37}$$

where we have used that events occuring in disjoint parts of space are independent, Hölder's inequality, and the fact that μ is translation invariant. Observing that $P[A\times\{0\} \to (0,T)] > 0$ whenever $A \cap \{-T,\ldots,T\}^{d-1} \neq \emptyset$, letting $N \to \infty$ for fixed T in (4.37), we see that

$$\limsup_{N\to\infty} \sup_{|B|\geq N} \int \mu(\mathrm{d}A) P\big[\eta_T^{A\times\{0\}} \cap B = \emptyset\big] \leq \int \mu(\mathrm{d}A) 1_{\{A \cap \{-T,\ldots,T\}^{d-1} = \emptyset\}}. \tag{4.38}$$

Letting $T \uparrow \infty$ the right-hand side converges to $\mu(\{\emptyset\})$, which is zero by assumption. ∎

Theorem 15 has the following corollary.

Corollary 19 (Homogeneous invariant laws) *If $\vec{\theta}'(p) > 0$, then the upper invariant law is the only homogeneous nontrivial invariant law of the oriented percolation process.*

Proof If $\vec{\theta}'(p) > 0$, then $P[-\infty \to (0,0)] = \vec{\theta}'(p) > 0$. Since the event $\{\exists (x,t) \in \mathbb{Z}^{d-1} \times \mathbb{Z} : -\infty \to (x,t)\}$ is translation invariant, we have $P[\exists (x,t) \in \mathbb{Z}^{d-1} \times \mathbb{Z} : -\infty \to (x,t)] = 1$ by ergodicity (compare the proof of Lemma 8), hence $P[\exists t \in \mathbb{Z} : \eta_t^{\{-\infty\}} \neq \emptyset] = 1$ and therefore also $P[\eta_0^{\{-\infty\}} \neq \emptyset] = 1$, so $\overline{\nu}$ is nontrivial. It is obviously is homogeneous as well. Now imagine that ν is another homogeneous nontrivial invariant law. Then by Theorem 15, the process started in $P[\eta_0 \in \cdot\,] = \nu$ satisfies $\nu = P[\eta_t \in \cdot\,] \Rightarrow \overline{\nu}$ as $t \to \infty$, so $\nu = \overline{\nu}$. ∎

4.3 Continuity of the percolation probability

The methods developed in the previous section can also be used to prove the analogue of Proposition 12 for oriented percolation. We start with a little lemma.

Lemma 20 (Occupied sites everywhere) *If* $\vec{\theta}'(p) > 0$, *then*

$$\lim_{N\to\infty} \sup_{|B|\geq N} P[\eta_0^{\{-\infty\}} \cap B = \emptyset] = 0. \tag{4.39}$$

Proof By Corollary 19, $\overline{\nu} = P[\eta_0^{-\infty} \in \cdot\,]$ is homogeneous and nontrivial. Therefore, by Lemma 18,

$$\lim_{t\to\infty} \limsup_{N\to\infty} \sup_{|B|\geq N} P[\eta_t^{-\infty} \cap B = \emptyset] = 0. \tag{4.40}$$

Thus, for each $\varepsilon > 0$ we can choose $t \geq 0$ such that $\limsup_{N\to\infty} \sup_{|B|\geq N} P[\eta_t^{-\infty} \cap B = \emptyset] \leq \varepsilon$. Since $P[\eta_0^{-\infty} \cap B = \emptyset] = P[\eta_t^{-\infty} \cap B = \emptyset]$, (4.39) follows. ∎

Proposition 21 (Continuity of the percolation probability) *The percolation probability* $\vec{\theta}'$ *is continuous on* $(\vec{p}_{\rm c}', 1]$.

Proof Right continuity of θ and $\vec{\theta}$ has been proved in Lemma 2, and the same proof works for $\vec{\theta}'$, so it suffices to prove left continuity. Let $\mathcal{A}(\mathbb{Z}^{d-1}\times\mathbb{Z})$ be our set of arrows and let $(U_a)_{a\in\mathcal{A}(\mathbb{Z}^{d-1}\times\mathbb{Z})}$ be independent, uniformly distributed $[0,1]$-valued random variables. Say that an arrow a is p-open if $U_a < p$. Then

$$\lim_{p'\uparrow p} \vec{\theta}'(p') = \lim_{p'\uparrow p} P[(0,0) \stackrel{p'}{\to} \infty] = P[\exists p' < p \ \text{ s.t. } \ (0,0) \stackrel{p'}{\to} \infty]. \tag{4.41}$$

Assuming that $p > \vec{p}_{\rm c}'$, we need to show that $(0,0) \stackrel{p}{\to} \infty$ implies $(0,0) \stackrel{p'}{\to} \infty$ for some $p' < p$. In order to do this, it suffices to show that there exists a $p'' < p$ and an (x,t) such that $(0,0) \stackrel{p}{\to} (x,t) \stackrel{p''}{\to} \infty$. For if this is the case, then, since all the arrows in the finite path $(0,0) \stackrel{p}{\to} (x,t)$ are p-open, the maximum of the U_a's along this path is smaller than p, which implies that this path is p'-open for some $p'' < p' < p$ and hence $(0,0) \stackrel{p'}{\to} \infty$.

Thus, we need to show that there exists a $p'' < p$ and an (x,t) such that $(0,0) \stackrel{p}{\to} (x,t) \stackrel{p''}{\to} \infty$. Pick any $\vec{p}_{\rm c}' < p'' < p$ and set

$$\xi_t^{\{\infty\}} := \{x \in \mathbb{Z}^{d-1} : (x,t) \stackrel{p''}{\to} \infty\}. \tag{4.42}$$

Then $\xi_t^{\{\infty\}}$ is distributed according to the upper invariant law with parameter p''. Note that for fixed $t \geq 0$, the sets $\eta_t^{\{(0,0)\}}$ and $\xi_t^{\{\infty\}}$ are independent. By a similar argument as in the proof of Theorem 15, using Lemma 17 and Lemma 20, we see that

$$\lim_{t\to\infty} P\big[\eta_t^{\{(0,0)\}} \cap \xi_t^{\{\infty\}} \neq \emptyset\big] = P[(0,0) \stackrel{p}{\to} \infty]. \tag{4.43}$$

This implies that

$$\lim_{t\to\infty} P\big[\exists x \text{ s.t. } (0,0) \stackrel{p}{\to} (x,t) \stackrel{p''}{\to} \infty \,\big|\, (0,0) \stackrel{p}{\to} \infty\big] = 1, \tag{4.44}$$

as required. ∎

5 Interacting particle systems

5.1 Phase transitions

The theory of percolation has a number of applications, especially to probabilistic problems in a setting where spatial structure is important. In particular, the theory has close links with the theory of interacting particle systems and phase transitions.

Stochastic systems consisting of a large number N of components often have the property that certain functions describing the properties of these systems develop singularities as N tends to infinity. A simple example is percolation. The probability that the origin of a large block $B_n = \{-n, n\}^d$ can be connected to the boundary ∂B_n along a path of open edges, is for each finite n an infinitely smooth, yes even analytic function of the percolation parameter p. Yet, in the limit $n \to \infty$, this function converges to $\theta(p)$, which does not have a first derivative at $p_{\rm c}$. (On the other hand, it is believed that outside $p_{\rm c}$, the function θ is analytic.) In the language of interacting particle systems, such a singularity, separating regions where the system behaves smoothly as a function of its parameter(s), is called a *phase transition.* It is believed that this is basically the same phenomenon as what is called a phase transition in daily use, e.g. the melting of ice, or the transition between the graphite and diamond form of carbon, which happen at precisely defined values of temperature, pressure, etc.

The theory of percolation can sometimes help in the study of these phase transitions, and of the phases that they separate. For example, the proof of the continuity of the percolation probability in unoriented percolation, based on the uniqueness of the infinite cluster, can be adapted to prove the continuity of the magnetisation in the Ising model, which is a simple model for a ferromagnet.

Oriented percolation also has a number of applications, in particular to models of particles that can disappear, but cannot spontaneously be created. Such models have applications in physics (nuclear chain reactions) and in theoretical biology (where particles represent organisms, or genes). In particular, *comparison with oriented percolation* is a technique that can often be applied to show that certain populations have a positive probability to survive. These sort of comparison arguments are also important within percolation theory itself. In particular, the proof by Bezuidenhout and Grimmett [BG90] that the oriented percolation probability is continuous at $p_{\rm c}$ is based on an argument that says that if oriented percolation survives, then it can compared with another oriented percolation process, with an arbitrarily high percolation parameter.

In the remainder of this section, we will have a look at two interacting particle systems that can be compared to oriented percolation.

5.2 Poisson point processes

Recall that an $\mathbb{N}$-valued random variable N is $\mathrm{Pois}(\lambda)$ distributed (with $\lambda \geq 0$) if

$$P[N = n] = \frac{\lambda^n}{n!} e^{-\lambda n} \qquad (n \geq 0). \tag{5.1}$$

A *Poisson point process* on the real line with *intensity* ρ is a random, locally finite subset $Q \subset \mathbb{R}$ such that

$$P\big[|Q \cap A| = n\big] = \frac{\rho\ell(A)^n}{n!} e^{-\rho\ell(A)n} \tag{5.2}$$

for all measurable $A \subset \mathbb{R}$ that have a finite Lebesgue measure $\ell(A)$, and such that $|Q \cap A_1|, \ldots, |Q \cap A_k|$ are independent whenever $A_1, \ldots, A_k$ are disjoint. For each $\rho \geq 0$, it is possible to construct such a Poisson point process with intensity ρ, and it is unique in distribution. One such construction goes as follows. If $\rho = 0$ we may set $Q = \emptyset$. Otherwise, let $(\sigma_k)_{k\geq 1}$ and $(\tau_k)_{k\geq 1}$ be independent exponentially distributed random variables with mean ρ^{-1}. Then

$$Q := \Big\{ -\sum_{k=1}^{n} \sigma_k : n \geq 1 \Big\} \cup \Big\{ \sum_{k=1}^{n} \tau_k : n \geq 1 \Big\} \tag{5.3}$$

is a Poisson point process with intensity ρ. It is a bit tedious to prove this, so we will skip the formal proof. Note however that if Q is a Poisson point process with intensity ρ, then

$$\begin{aligned} P\big[|Q \cap [t, t+\mathrm{d}t)| = 0\big] &= 1 - \rho\,\mathrm{d}t + O(\mathrm{d}t^2), \\ P\big[|Q \cap [t, t+\mathrm{d}t)| = 1\big] &= \rho\,\mathrm{d}t + O(\mathrm{d}t^2), \\ P\big[|Q \cap [t, t+\mathrm{d}t)| \geq 2\big] &= O(\mathrm{d}t^2) \end{aligned} \tag{5.4}$$

as $\mathrm{d}t \to 0$. Intuitively, we can think of Q as giving each infinitesimal interval $[t, t+\mathrm{d}t)$ an independent probability $\rho\,\mathrm{d}t$ to contain a point. Indeed, dividing the interval $[0, t)$ into n pieces $[\frac{k-1}{n}, \frac{k}{n})$ $(k = 1, \ldots, n)$, which independently of each other each have approximately probability $1 - \rho/n$ to have empty intersection with Q, we see that

$$P\big[Q \cap [0, t) = \emptyset\big] \cong \big(1 - \tfrac{\rho}{n}\big)^n \cong e^{-\rho t} \tag{5.5}$$

as $n \to \infty$. This gives at least a heuristic explanation of (5.3).

5.3 The contact process

For each $x \in \mathbb{Z}^d$, let Q_x be a Poisson point process with intensity 1. Likewise, for each $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$ with $|x - y| = 1$, let $Q_{x,y}$ be a Poisson point process with intensity λ, where $\lambda \geq 0$ is fixed. We assume that the Q_x and $Q_{x,y}$ are all independent. A usual way to depict a random realization of these Poisson processes is to draw $\mathbb{Z}^d$ horizontally, draw a time axis vertically, and mark each point $t \in Q_x$ by drawing an $*$ at the point (x, t), and mark each point $t \in Q_{x,y}$ by drawing an arrow from (x, t) to (y, t).

We can use the structure of random asterixes and arrows, created by these independent Poisson processes, to do something very similar to oriented percolation. Let us say that there is an *open path* from a point (x, s) to a point (y, t), denoted as $(x, s) \to (y, t)$, if there exist $s = t_0 \leq \cdots \leq t_n = t$ and $z_1, \ldots, z_n$ $(n \geq 1)$ such that $z_1 = x$, $z_n = y$, $Q_{z_k} \cap [t_{k-1}, t_k] =$ for all $k = 0, \ldots, n$, and $|z_{k+1} - z_k| = 1$ and $t_k \in Q_{z_k, z_{k+1}}$ for all $k = 1, \ldots, n-1$. Thus, an open path must walk upwards, may follow arrows, but must avoid asterixes. Setting

$$\eta_t^C := \{y \in \mathbb{Z}^d : C \to (y, t)\} \qquad (t \in \mathbb{R},\ C \subset \mathbb{Z}^d \times \mathbb{R}), \tag{5.6}$$

using the fact that Poisson processes in disjoint parts of space are independent, it is not hard to see that

$$P_t(A,\cdot\,) := P\big[\eta^{A\times\{s\}}_{s+t} \in \cdot\,\big] \qquad (t \geq 0,\ A \subset \mathbb{Z}^d) \tag{5.7}$$

defines a transition probability (in continuous time) on the space of all subsets of $\mathbb{Z}^d$. Any continuous-time Markov process $(\eta_t)_{t\geq 0}$ with these transition probabilities is called a *contact process* with *infection rate* λ. In particular, the process $(\eta^{A\times\{0\}}_t)_{t\geq 0}$ is such a contact process, started in the initial state A. We interpret the sites $x \in \mathbb{Z}^d$ as organisms (e.g., in dimension two, $\mathbb{Z}^2$ might be an infinite orchard), and we say that sites $x \in \eta_t$ are *infected* at time t, and all other sites are *healthy*. Note that each infected site has in each small time interval $[t, t+\mathrm{d}t)$ a probability $\mathrm{d}t$ to become healthy. Moreover, if x is infected and y is healthy, and $|x-y| = 1$, then x infects y with probability $\lambda\mathrm{d}t$. We describe this by saying that infectes sites become healthy with *rate* 1, and infect each healty site at distance one with *rate* λ.

The theory of the contact process (in dimension d) is more or less identical to that of oriented percolation (in dimension $d+1$). Almost all proofs that work for oriented percolation can immediately be adapted to work for the contact process, and vice versa. Indeed, most of the literature (e.g. the earlier mentioned [BG90, Har76]) actually treats the contact process; sometimes it is remarked that the same proof carries over to oriented percolation without a change.

There is one elementary point where this is perhaps not so obvious. In our proof that $\vec{p}_{\rm c} < 1$, we explicitly counted the number of contours blocking the origin from being connected to infinity.[3] It is not so obvious how to carry out this argument in the continuous setting of the contact process. Therefore, we will use a comparison argument to show that for sufficiently high λ, the contact process survives, i.e.,

$$P\big[\eta^{\{(0,0)\}}_t \neq \emptyset\ \forall t \geq 0\big] > 0. \tag{5.8}$$

It is easy to see that it suffices to prove this in one dimension (compare Lemma 4).

We will use comparison with oriented bond percolation on $\mathbb{Z}^2_{\rm even} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$. Thus we draw arrows from each point $(x,t) \in \mathbb{Z}^2_{\rm even}$ to the points $(x-1, t+1)$ and $(x+1, t+1)$, and let arrows, independently of each other, be open with probability p. We write

$$\xi_t := \{x \in \mathbb{Z} : C \to (0,0)\} \qquad (t \in \mathbb{Z}). \tag{5.9}$$

Proposition 22 (Comparison with oriented percolation) *Fix $\varepsilon > 0$. Let $(\eta_t)_{t\geq 0}$ be a one-dimensional contact process with infection rate λ, started in the initial state $\eta_0 = \{0\}$. Let $(\xi_t)_{t\geq 0}$ be the oriented bond percolation process defined in (5.9) with parameter*

$$p = \big(1 - \big(1 - e^{-2\varepsilon}\big(1 - e^{-\lambda\varepsilon}\big)\big)^{1/3}\big)^2 \tag{5.10}$$

If $p \geq \frac{1}{4}$, then $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ can be coupled such that $\eta_{\varepsilon t} \supset \xi_t$ for all integer $t \geq 1$.

[3] On the other hand, it is easy to show that if $\lambda \leq 1$, the contact process dies out. This is because the number of infected sites can be bounded from above by a critical branching process. This is a sort of analogue of Lemma 5 for the contact process.

We have proved in Proposition 7 that ξ survives with positive probability if $p > 8/9$. Since for λ large enough, by choosing ε appropriately (e.g. $\varepsilon = \lambda^{-1/2}$), we can make the expression in (5.10) larger than 8/9, it follows that the contact process survives with positive probability if λ is large enough.

The bound on the *critical infection rate* λ_c (which is defined analogously to p_c) that follows from this argument is not particularly good. A proof (along different lines) that $\lambda_c \leq 2$ can be found in [Lig85].

Proof of Proposition 22 Given the Poisson point processes used in the construction of the contact process, we can construct an oriented percolation model as follows: we let the arrow from (x,t) to $(y,t+1)$, where $y = x \pm 1$, be open if $Q_x \cap [t\varepsilon, (t+1)\varepsilon] = \emptyset$, $Q_y \cap [t\varepsilon, (t+1)\varepsilon] = \emptyset$, and $Q_{x,y} \cap [t\varepsilon, (t+1)\varepsilon] \neq \emptyset$, i.e., if there is an infection from x to y and neither x nor y recovers during the time interval $[t\varepsilon, (t+1)\varepsilon]$; otherwise we let this arrow be closed. Then arrows are open with probability

$$p' := e^{-2\varepsilon}\big(1 - e^{-\lambda\varepsilon}\big). \tag{5.11}$$

Moreover, if $(x,t) \in \mathbb{Z}^2_{\text{even}}$ can be reached from the origin along a path of open arrows, then $x \in \eta_{\varepsilon t}$. The problem is that arrows are not independent. They are, however, 3-dependent in the sense of Theorem 23 below, so we may estimate the oriented percolation model we have just constructed from below by an oriented percolation process with independent arrows with the parameter from (5.10). ∎

5.4 K-dependence

In order to complete the proof of Proposition 22, we need a technical result on K-dependence. Let Λ be a countable set and let $(\chi_x)_{x\in\Lambda}$ be a collection of random variables. We will say[4] that the $(\chi_x)_{x\in\Lambda}$ are K-dependent if for each $x \in \Lambda$ there exists a $x \in \Delta_x \subset \Lambda$ with $|\Delta_x| \leq K$ such that

$$\chi_x \text{ is independent of } (\chi_y)_{y\in\Lambda\setminus\Delta_x} \tag{5.12}$$

for all $x \in \Lambda$.

Theorem 23 (Domination of product measures) *Let Λ be a countable set and let $(\chi_x)_{x\in\Lambda}$ be K-dependent $\{0,1\}$-valued random variables with $P[\chi_x = 1] \geq p$ for all $x \in \Lambda$. Then it is possible to couple $(\chi_x)_{x\in\Lambda}$ to a collection of independent $\{0,1\}$-valued random variables $(\tilde\chi_x)_{x\in\Lambda}$ with*

$$P[\tilde\chi_x = 1] = \tilde p := \big(1 - (1-p)^{1/K}\big)^2 \tag{5.13}$$

in such a way that $\tilde\chi_x \leq \chi_x$ for all $x \in \Lambda$.

Proof Since we can always choose some arbitrary denumeration of Λ, we may assume that $\Lambda = \mathbb{N}$. Our strategy will be as follows. We will choose $\{0,1\}$-valued random variables $(\psi_x)_{x\in\Lambda}$ with $P[\psi_x = 1] = r$, independent of each other and of the $(\chi_x)_{x\in\mathbb{N}}$, and put

$$\chi'_x := \psi_x \chi_x \qquad (x \in \mathbb{N}). \tag{5.14}$$

[4]Our definition deviates a bit from the usual definition, which applies to the case $\Lambda = \mathbb{Z}^d$ only. See e.g. [Lig99].

Note that the $(\chi'_x)_{x\in\mathbb{N}}$ are a 'thinned out' version of the $(\chi_x)_{x\in\mathbb{N}}$. In particular, $\chi'_x \leq \chi_x$ $(x \in \mathbb{N})$. We will show that for an appropriate choice of r,

$$P[\chi'_n = 1 \,|\, \chi'_0, \ldots, \chi'_{n-1}] \geq \tilde{p} \tag{5.15}$$

for all $n \geq 0$, and we will show that this implies that the $(\chi'_x)_{x\in\mathbb{N}}$ can be coupled to independent $(\tilde{\chi}_x)_{x\in\Lambda}$ as in (5.13) in such a way that $\tilde{\chi}_x \leq \chi'_x \leq \chi_x$ $(x \in \mathbb{N})$.

We start with the latter claim. Imagine that (5.15) holds. Set

$$p'_n(\varepsilon_0, \ldots, \varepsilon_{n-1}) := P[\chi'_n = 1 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \tag{5.16}$$

whenever $P[\chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] > 0$. Let $(U_n)_{n\in\mathbb{N}}$ be independent, uniformly distributed $[0,1]$-valued random variables. Set

$$\tilde{\chi}_n := 1_{\{U_n > \tilde{p}\}} \qquad (n \in \mathbb{N}) \tag{5.17}$$

and define inductively

$$\chi'_n := 1_{\{U_n > p'_n(\chi'_0, \ldots, \chi'_{n-1})\}} \qquad (x \in \mathbb{N}). \tag{5.18}$$

Then

$$P[\chi'_n = \varepsilon_n, \ldots, \chi'_0 = \varepsilon_0] = p_n(\varepsilon_0, \ldots, \varepsilon_{n-1}) \cdots p_0. \tag{5.19}$$

This shows that these new χ'_n's have the same distribution as the old ones, and they are coupled to $\tilde{\chi}_x$'s as in (5.13) in such a way that $\tilde{\chi}_x \leq \chi'_x$.

What makes life complicated is that (5.15) does not always hold for the original $(\chi_x)_{x\in\mathbb{N}}$, which is why we have to work with the thinned variables $(\chi'_x)_{x\in\mathbb{N}}$.[5] We observe that

$$P[\chi'_n = 1 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] = rP[\chi_n = 1 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}]. \tag{5.20}$$

We will prove by induction that for an appropriate choice of r,

$$P[\chi_n = 0 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \leq 1 - r. \tag{5.21}$$

Note that this is true for $n = 0$ provided that $r \leq p$. Let us put

$$\begin{aligned} E_0 &:= \{x \in \Delta_x : 0 \leq x \leq n-1,\ \varepsilon_x = 0\}, \\ E_1 &:= \{x \in \Delta_x : 0 \leq x \leq n-1,\ \varepsilon_x = 1\}, \\ F &:= \{x \notin \Delta_x : 0 \leq x \leq n-1\}. \end{aligned} \tag{5.22}$$

[5]Indeed, let $(\phi_n)_{n\geq 0}$ be independent $\{0,1\}$-valued random variables with $P[\phi_n = 1] = \sqrt{p}$ for some $p < 1$, and put $\chi_n := \phi_n\phi_{n+1}$. Then the $(\chi_n)_{n\geq 0}$ are 1-dependent with $P[\chi_n = 1] = p$, but $P[\chi_n = 1|\chi_{n-1} = 0, \chi_{n-2} = 1] = 0$.

Then

$$\begin{aligned}
&P[\chi_n = 0 \,|\, \chi'_0 = \varepsilon_0, \ldots, \chi'_{n-1} = \varepsilon_{n-1}] \\
&\quad = P\big[\chi_n = 0 \,\big|\, \chi'_x = 0 \ \forall x \in E_0,\ \chi_x = 1 = \psi_x \ \forall x \in E_1,\ \chi'_x = \varepsilon_x \ \forall x \in F\big] \\
&\quad = P\big[\chi_n = 0 \,\big|\, \chi'_x = 0 \ \forall x \in E_0,\ \chi_x = 1 \ \forall x \in E_1,\ \chi'_x = \varepsilon_x \ \forall x \in F\big] \\
&\quad = \frac{P\big[\chi_n = 0,\ \chi'_x = 0 \ \forall x \in E_0,\ \chi_x = 1 \ \forall x \in E_1,\ \chi'_x = \varepsilon_x \ \forall x \in F\big]}{P\big[\chi'_x = 0 \ \forall x \in E_0,\ \chi_x = 1 \ \forall x \in E_1,\ \chi'_x = \varepsilon_x \ \forall x \in F\big]} \\
&\quad \leq \frac{P\big[\chi_n = 0,\ \chi'_x = \varepsilon_x \ \forall x \in F\big]}{P\big[\psi_x = 0 \ \forall x \in E_0,\ \chi_x = 1 \ \forall x \in E_1,\ \chi'_x = \varepsilon_x \ \forall x \in F\big]} \\
&\quad = \frac{P\big[\chi_n = 0 \,\big|\, \chi'_x = \varepsilon_x \ \forall x \in F\big]}{P\big[\psi_x = 0 \ \forall x \in E_0,\ \chi_x = 1 \ \forall x \in E_1 \,\big|\, \chi'_x = \varepsilon_x \ \forall x \in F\big]} \\
&\quad \leq \frac{1-p}{(1-r)^{|E_0|} P\big[\chi_x = 1 \ \forall x \in E_1 \,\big|\, \chi'_x = \varepsilon_x \ \forall x \in F\big]} \leq \frac{1-p}{(1-r)^{|E_0|}\, r^{|E_1|}},
\end{aligned} \tag{5.23}$$

where in the last step we have used K-dependence and the fact that

$$P\big[\chi_x = 1 \ \forall x \in E_1 \,\big|\, \chi'_x = \varepsilon_x \ \forall x \in F\big] \geq r^{|E_1|}. \tag{5.24}$$

We claim that (5.24) is a consequence of the induction hypothesis (5.21). Indeed, we may assume that the induction hypothesis (5.21) holds regardless of the ordering of the first n elements, so without loss of generality we may assume that $E_1 = \{n-1, \ldots, m\}$ and $F = \{m-1, \ldots, 0\}$, for some m. Then the left-hand side of (5.24) may be written as

$$\begin{aligned}
&\prod_{k=m}^{n-1} P\big[\chi_k = 1 \,\big|\, \chi_x = 1 \ \forall m \leq x < k,\ \chi'_x = \varepsilon_x \ \forall 0 \leq x < m\big] \\
&\quad = \prod_{k=m}^{n-1} P\big[\chi_k = 1 \,\big|\, \chi'_x = 1 \ \forall m \leq x < k,\ \chi'_x = \varepsilon_x \ \forall 0 \leq x < m\big] \geq r^{n-m}.
\end{aligned} \tag{5.25}$$

If we assume moreover that $r \geq \frac{1}{2}$, then $r^{|E_1|} \geq (1-r)^{|E_1|}$ and therefore the right-hand side of (5.23) can be further estimated as

$$\frac{1-p}{(1-r)^{|E_0|}\, r^{|E_1|}} \leq \frac{1-p}{(1-r)^{|\Delta_n \cap \{0,\ldots,n-1\}|}} \leq \frac{1-p}{(1-r)^{K-1}}. \tag{5.26}$$

We see that in order for our proof to work, we need $\frac{1}{2} \leq r \leq p$ and

$$\frac{1-p}{(1-r)^{K-1}} \leq 1 - r. \tag{5.27}$$

In particular, choosing $r = 1 - (1-p)^{1/K}$ yields equality in (5.27). Having proved (5.21), we see by (5.20) that (5.15) holds provided that we put $\tilde{p} := r^2$. ∎

5.5 Parity preserving branching

In this section, we will have a look at another interacting particle system for which one can prove results by comparison with oriented percolation. Processes of the type we will consider have recently been studied by Neuhauser and Pacala [NP99] and several other authors. They are relevant in evolution biology for studying competing, closely related species. The model with the precise description below was introduced during work in progress of me with Anja Sturm (Delaware).

Our process of interest will be a continuous-time Markov process $(X_t)_{t\geq 0}$ taking values in the space $\{0,1\}^{\mathbb{Z}}$, with the following description. If $X_t(x) = 1$, then we say that the site $x \in \mathbb{Z}$ is at time $t \geq 0$ occupied by a particle; otherwise it is vacant. With rate $1-\beta$ (i.e., according to a Poisson process with this intensity), the particle jumps one site to the right, with the rule that if there is already another particle present on that site, the particles annihilate (i.e., both particles disappear). Likewise, also with rate $1-\beta$, the particle jumps one site to the left, again annihilating with any other particle that might be present there. With rate β, the particle creates two new particles on the two sites immediately to the right of it. If any of these newly created particles land on a site that is already occupied, they immediately annihilate with the particle there. Likewise, also with rate β, the particle creates two new particles on the two sites immediately to the left of it, again annihilating with any other particle that might be present there. We call β the *branching rate* and we call X a *parity preserving branching process.*

A peculiar property of the model is that it preserves parity, i.e., if we start the model in an initial state X_0 with a (finite) even number of particles, then the number of particles will stay even for all time, and likewise if we start with an odd number, it stays odd. If the branching rate β is zero, then our model consists of *annihilating random walks.* In this case, it is easy to prove that if one starts the process with an even number of particles, eventually all particles will annihilate each other, while if one starts with an odd number of particles, eventually all that is left is a single particle, that cannot be annihilated anymore.

Computer simulations suggest that the process goes through a phase transition as the branching rate β is increased. If β is below some critical value, say β_{c}, then the process started in an even initial state dies out, while the process started in an odd initial state stays bounded in the sense that it is infinitly often reduced to just one particle. On the other hand, if $\beta > \beta_{\mathrm{c}}$, the process started in an even initial state has a positive chance to survive, and whenever the process (even or odd) survives, the number of particles tends to infinity.

Surprisingly little of this has been rigorously proved. For example, it is not even known whether survival is monotone in the parameter β, i.e., it cannot be excluded that the (even) process survives with positive probability for some β, while it dies out for some $\beta' > \beta$. The difficulty is that because of the annihilation, more particles in the initial state may mean less particles at some later time, so the usual coupling arguments do not work. The parity preservation causes additional difficulties. For example, it is not known if survival implies that the number of particles tends to infinity. In particular, the proof of Lemma 17 does not work in the present setting since one cannot exclude the scenario that the process survives by having most of the time just two particles at positions very far from each other. In general, parity preservation makes extinction harder, and as a result, it it not proved that the process shoud get extinct for any $\beta > 0$ (i.e., it is not proved that $\beta_c > 0$).

In spite of all these difficulties, we will see that by a clever comparison with oriented percolation, we can at least prove that for β sufficiently close to 1, the process survives, and the number of particles tends to infinity. We start by proving that the process survives if $\beta = 1$.

Lemma 24 (Survival for $\beta = 1$) *If $\beta = 1$ and X is started in an even nonzero initial state X_0, then $P[X_t \neq \emptyset\ \forall t \geq 0] > 0$.*

Proof We will show that the position of the right-most particle tends to move to the right, while the position of the left-most particle tends to move to the left. As a result, there is a possibility that the two never meet, and hence the process survives. By symmetry, it suffices to show that the position of the right-most particle tends to move to the right.

Let n be the position of the right-most particle at a certain time t. Then certainly, $X_t(n+1) = 0$ and $X_t(n+2) = 0$. Imagine for the moment that also $X_t(n-1) = 0$ and $X_t(n-2) = 0$, i.e.,

$$\big(X_t(n-2), X_t(n-1), X_t(n), X_t(n+1), X_t(n+2)\big) = (0,0,1,0,0). \tag{5.28}$$

Then during a small time interval $[t, t+\mathrm{d}t)$, there is approximately a probability $\beta\mathrm{d}t = 1\cdot\mathrm{d}t$ that the Poisson process governing the branching to the right becomes active, and likewise for the Poisson process governing the branching to the left. Therefore, at time $t+\mathrm{d}t$, we see that

$$\begin{aligned}&\big(X_{t+\mathrm{d}t}(n-2), X_{t+\mathrm{d}t}(n-1), X_{t+\mathrm{d}t}(n), X_{t+\mathrm{d}t}(n+1), X_{t+\mathrm{d}t}(n+2)\big)\\ &\qquad = \begin{cases} (1,1,1,0,0) & \text{with probability } \mathrm{d}t,\\ (0,0,1,1,1) & \text{with probability } \mathrm{d}t.\end{cases}\end{aligned} \tag{5.29}$$

There may also be changes in the values of $X_{t+\mathrm{d}t}(n-2)$ and $X_{t+\mathrm{d}t}(n-1)$ due to branching of particles further to the left, but these events do not change the position of the right-most particle. In fact, branching to the left is also irrelevant for the position of the right-most particle, so it is only the transition

$$00100 \mapsto 00111 \quad \text{with rate } 1 \tag{5.30}$$

that is relevant to us. Note that if the configuration around the right-most particle is 00100, then the position of the right-most particle can only increase. We now go through the different possible configurations around the right-most particle one by one, and write down only those transitions that change the position of the right-most particle. We see that

$$01100 \mapsto \begin{cases} 01010 & \text{with rate } 1,\\ 01111 & \text{with rate } 1,\end{cases} \tag{5.31}$$

resulting in an average increase of the right-most particle during the time interval $[t, t+\mathrm{d}t)$ of $\frac{3}{2}\mathrm{d}t$. Further,

$$10100 \mapsto \begin{cases} 11000 & \text{with rate } 1,\\ 10111 & \text{with rate } 1,\end{cases} \tag{5.32}$$

resulting in an average increase of $\mathrm{d}t$, an

$$11100 \mapsto \begin{cases} 10000 & \text{with rate } 1, \\ 11010 & \text{with rate } 1, \\ 11111 & \text{with rate } 1, \end{cases} \tag{5.33}$$

resulting in an average increase of $\mathrm{d}t$. We conclude that no matter what the configuration around the right-most particle looks like, the average increase of the right-most particle during the time interval $[t, t+\mathrm{d}t)$ is always at least $1 \cdot \mathrm{d}t$. We express this by saying that the right-most particle has a *drift* of at least 1 to the right.

There are now standard ways for showing that there is a positive probability that the right-most and the left-most particle never meet, and hence the process survives.

(For example, if D_t is the distance between the left-most and the right-most particle at time t, then the calculations above can be used to show that the function θ^{D_t} decreases on average, i.e., is a supermartingale, provided that $\theta^2 + \theta + \theta^{-2} \leq 3$ and $\theta^2 + \theta^{-1} \leq 2$. This is true for some $\theta \in (0, 1)$ sufficiently close to 1, and as a result the probablility that the process survives can be estimated from below by $1 - \theta^{D_0}$.) ∎

The proof of Lemma 24 breaks down if $\beta < 1$, since in this case, one particle may jump on top of the other particle, which yields the transition

$$01100 \mapsto 00000 \quad \text{with rate } 2(1-\beta), \tag{5.34}$$

leading to a decrease of the position of the right-most particle that can be arbitrarily large. We therefore need to use percolation theory.

Theorem 25 (Comparison with oriented percolation) *For each $p < 1$ there exist $\beta' < 1$, $L \geq 1$, and $T > 0$ such that for each $\beta \geq \beta'$, the parity preserving branching process X with branching rate β, started in an arbitrary initial state X_0, can be coupled to oriented bond percolation on $\mathbb{Z}^2_{\text{even}}$ with parameter p, in such a way that*

$$G := \big\{(k, s) \in \mathbb{Z}^2_{\text{even}} : X_{sT} \cap \{2Lk - L, \ldots, 2Lk + L\} \neq \emptyset\big\} \tag{5.35}$$

satisfies

$$G \ni (k, 0) \to (l, s) \text{ implies } (l, s) \in G. \tag{5.36}$$

Proof It suffices to prove (5.36) for $s = 1$; the general statement then follows by induction. Thus, we need to show that we can make arrows between points $(k, 0)$ and $(k \pm 1, 1)$ independently open or closed, in such a way that if $X_0 \cap \{2Lk - L, \ldots, 2Lk + L\} \neq \emptyset$ and the arrow from $(k, 0)$ to $(l, 1)$ is open, where $l = k+1$ or $k-1$, then also $X_T \cap \{2Ll - L, \ldots, 2Ll + L\} \neq \emptyset$.

Our strategy will be to first make arrows open or closed in a K-dependent way, such that each arrow has a very large probability to be open, and then use Theorem 23 to estimate these arrows from below by arrows that are independently open with a new probability p that is still large.

We will first do the argument for $\beta = 1$, and then see how our arguments can be adapted to cover also the case that β is sufficiently close to one.

Let us first try to prove that if $X_0 \cap \{-L, \ldots, L\} \neq \emptyset$, then, for suitable L and T, with large probability also $X_L \cap \{2L - L, \ldots, 2L + L\} \neq \emptyset$.

In the proof of Lemma 24, we have seen that the right-most particle of a parity preserving branching process (or in fact any particle with at least two empty spaces to the right of it) has a drift of at least $+1$ to the right. Since our aim is to 'steer' at least one particle into the interval $\{2L-L, \ldots, 2L+L\}$, we are more interested in the particle that is closest to the point $2L$ than in the absolutely right-most particle. In the same way as in the proof of Lemma 24, we see that as long as the particle that is closest to the point $2L$ is at least two positions away from the point $2L$, it has a drift of at least 1 in the direction of $2L$. Since we are assuming that there is initially a particle somewhere in $\{-L, \ldots, L\}$, the particle that is closest to the point $2L$ is initially at most a distance $3L$ from its aim. Therefore, by some strong law of large numbers, we expect this particle to have reached its aim by time T, if T is at least $3L$, and L is very large. Since the particle feels a drift towards $2L$ as soon as it is more than two positions away from it, we expect it to stay in the vicinity of $2L$ for a long time.

All of this can be made rigorous. Let R_t be the position of the particle that is closest to the point $2L$ at time t. With the help of large deviation theory, we can get precise estimates on the probability that R_t reaches $2L$ before the time T, and then stays close to it for the remaining time. Set

$$\tau := \inf\{t \geq 0 : |R_t - 2L| \leq 2\}. \tag{5.37}$$

Then large deviation theory tells us that

$$P[\tau \geq (3+s)L] \leq C(s)e^{-\lambda(s)L} \qquad (s > 0), \tag{5.38}$$

for some $C < \infty$ and $\lambda(s) > 0$. Likewise, if we start with $|R_0 - 2L| \leq 2$, and put

$$\tau' := \inf\{t \geq 0 : |R_t - 2L| > L\}, \tag{5.39}$$

then it can be shown that

$$P[\tau' \leq sL] \leq C'(s)Le^{-\lambda'(s)L} \qquad (s \geq 0) \tag{5.40}$$

for some $C' < \infty$ and $\lambda'(s) > 0$. In particular, if we choose $T = 4L$, the large deviation estimates (5.38) and (5.40) tell us that the probability that the particle closest to $2L$ comes within a distance 2 of the point $2L$ and then stays in the interval $\{L, \ldots, 3L\}$ for the rest of the interval $[0, T]$ can be bounded from below by

$$1 - C(1)e^{-\lambda(1)} - C'(4)Le^{-\lambda'(4)L}. \tag{5.41}$$

This can be made arbitrarily close to 1 by choosing L large enough.

In view of this, we are tempted to make the following definition: If for given $(k,0), (l,1) \in \mathbb{Z}^2_{\text{even}}$ with $l = k \pm 1$, one has $X_0 \cap \{2Lk - L, \ldots, 2Lk + L\} \neq \emptyset$, then we say that there is an open arrow from $(k, 0)$ to $(l, 1)$ provided that the particle that stays closest to the point $2Ll$ reaches that point before time T and then stays in the interval $\{2Ll - L, \ldots, 2Ll + L\}$ for the rest of the interval $[0, T]$. If $X_0 \cap \{2Lk - L, \ldots, 2Lk + L\} = \emptyset$, then we always make the arrow from $(k, 0)$ to $(l, 1)$ open.

With this definition, arrows are open with high probability, and if $X_0 \cap \{2Lk-L, \ldots, 2Lk+L\} \neq \emptyset$ and the arrow from $(k,0)$ to $(l,1)$ is open, then also $X_T \cap \{2Ll-L, \ldots, 2Ll+L\} \neq \emptyset$. Unfortunately, the arrows in this definition are not K-dependent for any finite K. Therefore, we have to modify our definition a little bit.

We recall that we have constructed our process using independent Poisson point processes. Thus, for each point $n \in \mathbb{Z}$ there is a Poisson point processes $Q_n^{\mathrm{b}+}$ with intensity 1 on the real line, such that at each time $t \in Q_n^{\mathrm{b}+}$, if there is a particle in n at that time, this particle branches and produces two new particles on the right of it. Likewise, there is a Poisson point processes $Q_n^{\mathrm{b}-}$ for branching to the left. We depict a point $t \in Q_n^{\mathrm{b}+}$ by drawing two arrows, one from (n,t) to $(n+1,t)$ and one from (n,t) to $(n+2,t)$. Because of the annihilation, it is *not* true that whenever there is a particle at n at time 0, and there is a path along arrows from $(n,0)$ to (m,t), then there must also be a particle at m at time t. However, it is still true that if there is *no* path from $(n,0)$ to (m,t), then if we want to decide whether there is a particle at m at time t, it is *irrelevant* whether there was a particle at n at time 0 or not. Thus, *upward paths along arrows* give us information about *flow of information.*

We observe that the right-most upward path, using arrows of the Poisson point processes $Q^{\mathrm{b}+}$ and $Q^{\mathrm{b}-}$, starting in a given set, has a drift $+2$ to the right. Therefore, during the time interval $[0,T]$, we expect such a path to cover roughly a distance $2T$. In view of this, we make the following definition: We choose $T = 4L$. If for given $(k,0),(l,1) \in \mathbb{Z}^2_{\mathrm{even}}$ with $l = k \pm 1$, one has $X_0 \cap \{2Lk-L, \ldots, 2Lk+L\} = \emptyset$, then we always make the arrow from $(k,0)$ to $(l,1)$ open. If $X_0 \cap \{2Lk-L, \ldots, 2Lk+L\} \neq \emptyset$, then we say that there is an open arrow from $(k,0)$ to $(l,1)$ provided that: 1. $X_{4L} \cap \{2Ll-L, \ldots, 2Ll+L\} \neq \emptyset$, and: 2. There is no upward path through the Poisson point processes $Q^{\mathrm{b}+}$ and $Q^{\mathrm{b}-}$ ending in $\{2Ll-L, \ldots, 2Ll+L\}$ and starting outside $\{2Ll-L-9L, \ldots, 2Ll+L+9L\}$. Using large deviation theory, the probability that an arrow is open can be bounded from below by an expression of the form

$$1 - C(1)e^{-\lambda(1)} - C'(4)Le^{-\lambda'(4)L} - C''e^{-\lambda''L}, \tag{5.42}$$

where the last term estimates the probability that there is a path through the Poisson point processes ending in $\{2Ll-L, \ldots, 2Ll+L\}$ and starting outside $\{2Ll-L-9L, \ldots, 2Ll+L+9L\}$. By choosing L large enough, the probability in (5.42) can be made arbitrarily close to one. Moreover, since there is no information coming in from outside $\{2Ll-L-9L, \ldots, 2Ll+L+9L\}$, arrows are K-dependently open or closed for a suitable K, which does not depend on L. (In fact, $K = 22$ is OK.)

This completes the proof in the case $\beta = 1$. In order to adapt the argument to β sufficiently close to 1, only small changes are necessary. By a small change in the speed of time, we may look at the case that branching (to the left or right) happens with rate 1, and in addition particles jump to the left ot right with probability ε, where ε is small. Adapting our earlier definition, in order for an arrow from $(k,0)$ to $(l,1)$ to be open, we require in addition that: 3. The Poisson point processes for the jump events are empty on $\{2Ll-L-9L, \ldots, 2Ll+L+9L\}$ during the time interval $[0,4L]$. Then, first choosing L large enough and then ε small enough, the probability that an arrow is open can be made as close to one as one wishes, and arrows are K-dependent with the same K as before. ∎

Remarks The value of β' in Theorem 25 that we obtain from the proof is horrendeously bad, so it is only the qualitative statement that there exists such a β' that is of interest. The proof follows a general pattern explained in [Dur91], although the latter reference is rather sloppy when it comes to proving K-dependence. The title of that reference ('A new method for proving the existence of phase transitions') is also somewhat misleading, since comparison with oriented percolation had been used long before then. It was only around 1990, however, that people found out how to use these arguments for processes that do not have nice monotonicity properties (in the language of interacting particle systems: non-attractive models).

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