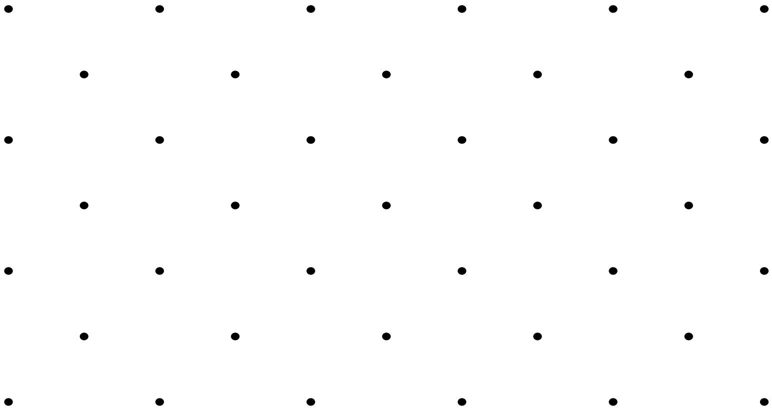
The Brownian web and net

Jan M. Swart

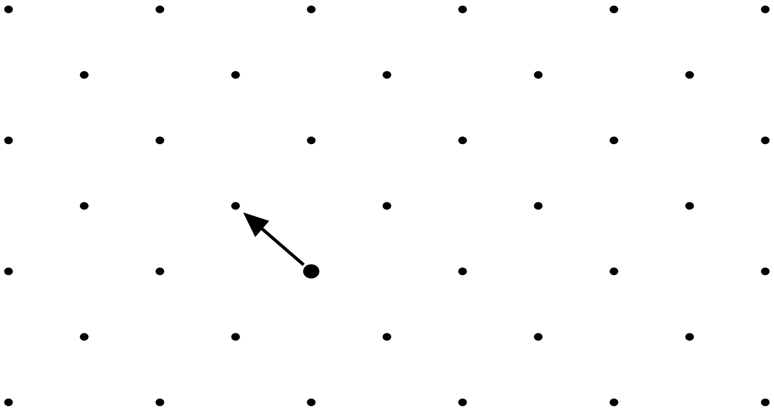
Bedlewo, April 22, 2024

The even sublattice



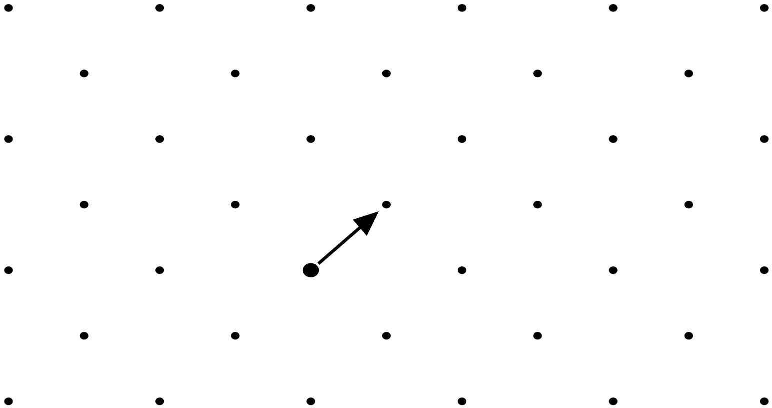
$$\mathbb{Z}^2_{\mathrm{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}.$$

Arrow configurations



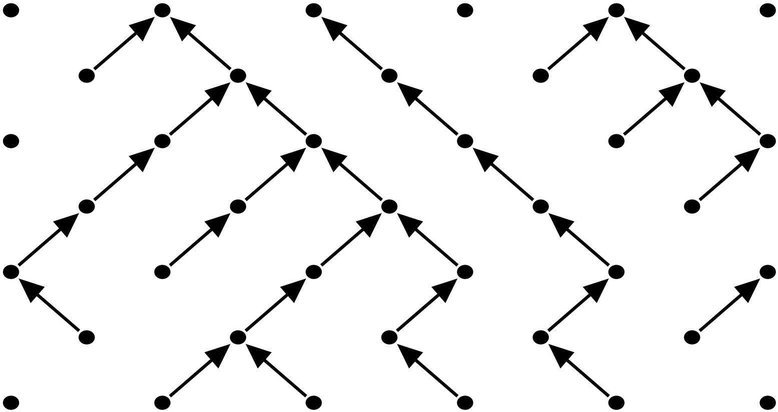
With probability $\frac{1}{2}$ we draw an arrow to the left...

Arrow configurations



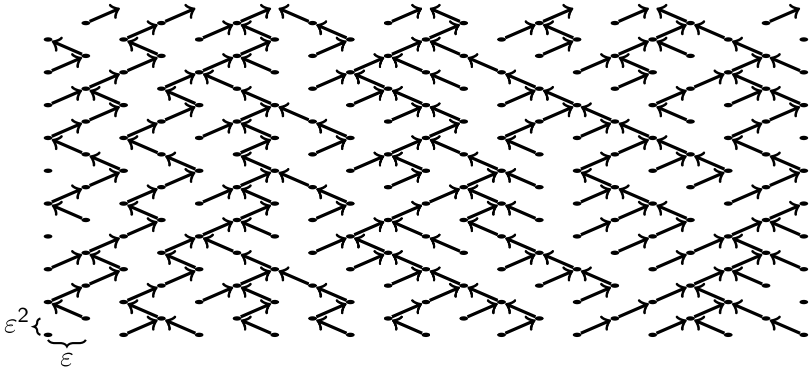
and with probability $\frac{1}{2}$ we draw it to the right.

Arrow configurations



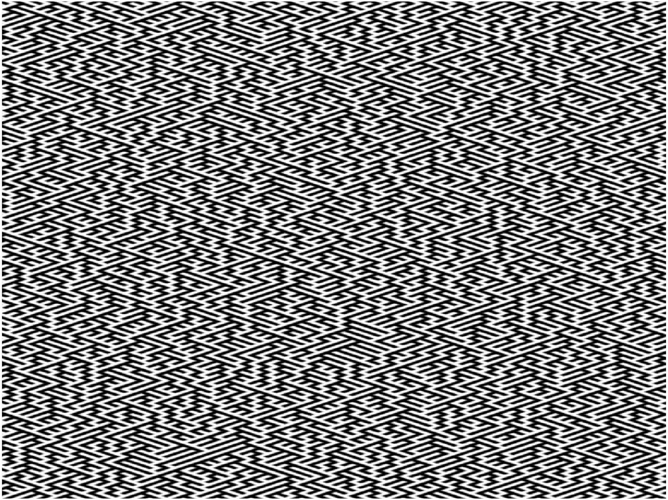
Independently for each space-time point

Scaling limit



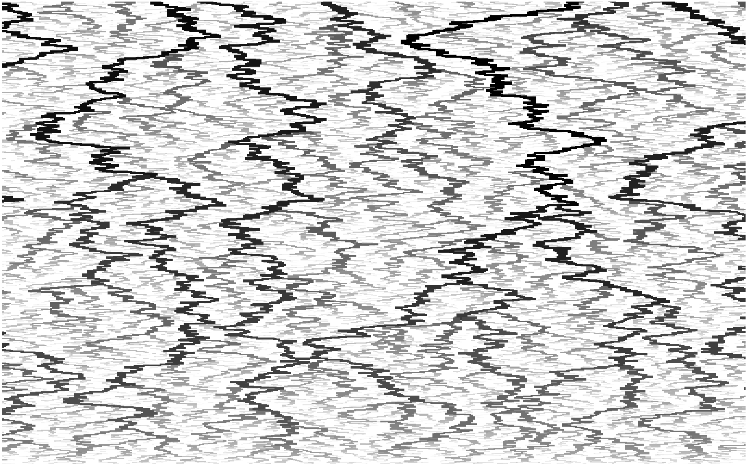
We rescale diffusively, multiplying all spatial distances with ε and all temporal distances with ε^2.

Scaling limit



Our aim is to describe the limit as $\varepsilon \to 0$.

Scaling limit



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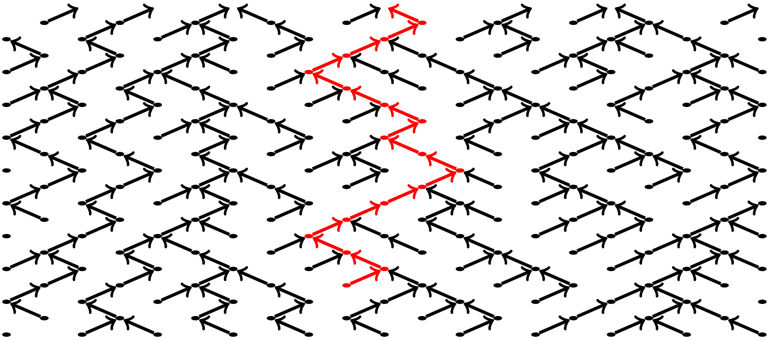
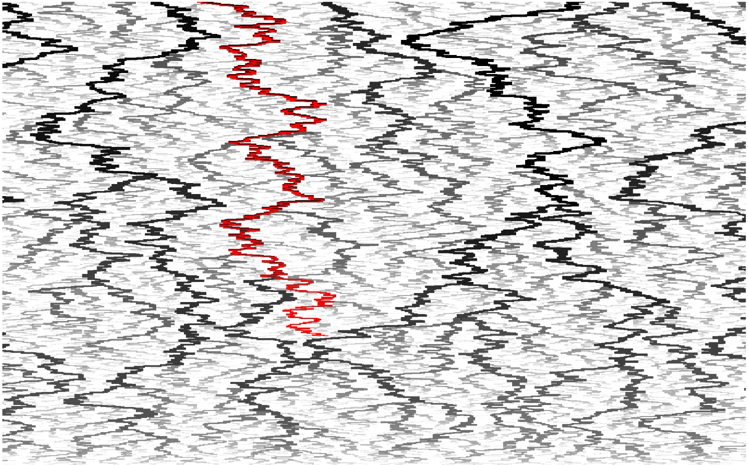
Paths



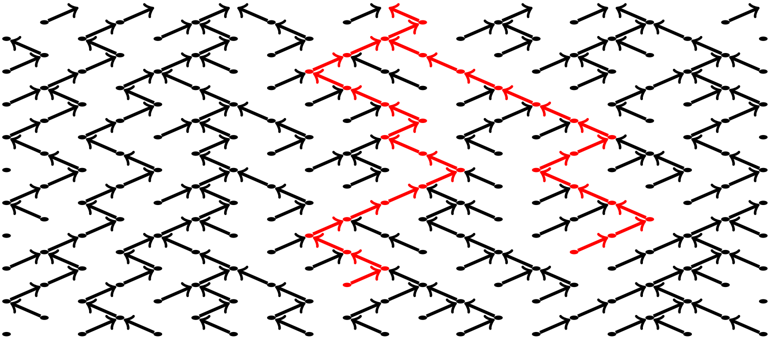
Each space-time point is the starting point of a random walk path.

Paths



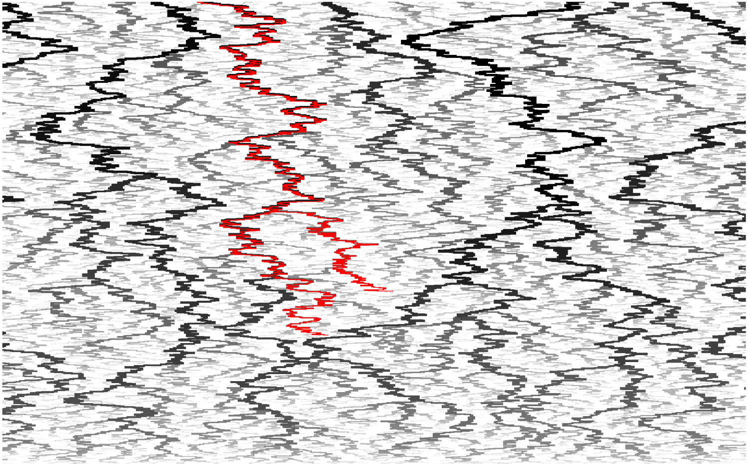
The diffusive scaling limit of a single random walk
is a Brownian motion.

Paths



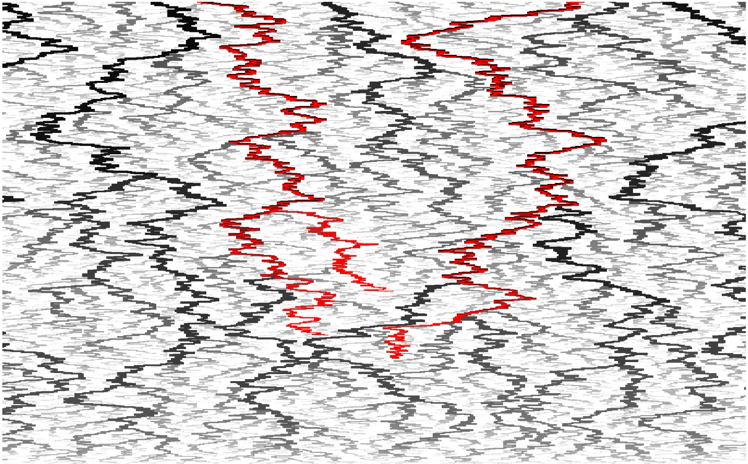
Random walk paths started at different space-time points *coalesce* as soon as they meet.

Paths



In the limit, we obtain *coalescing Brownian motions*.

Paths



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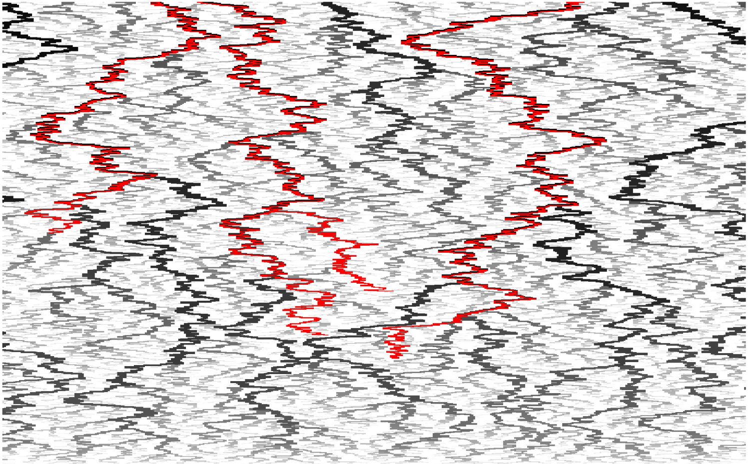

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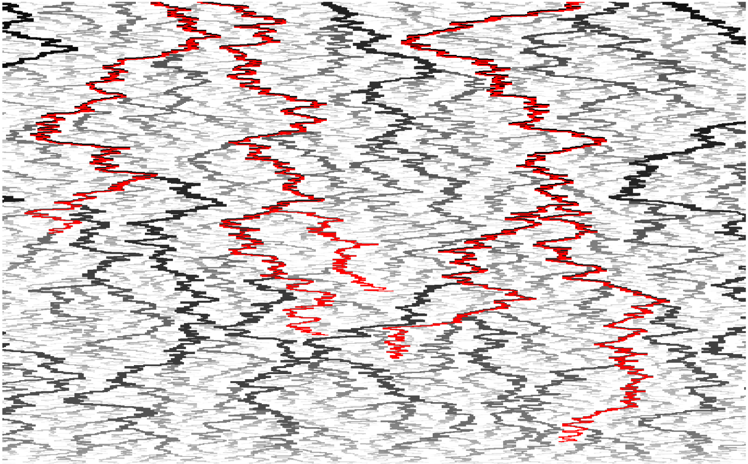

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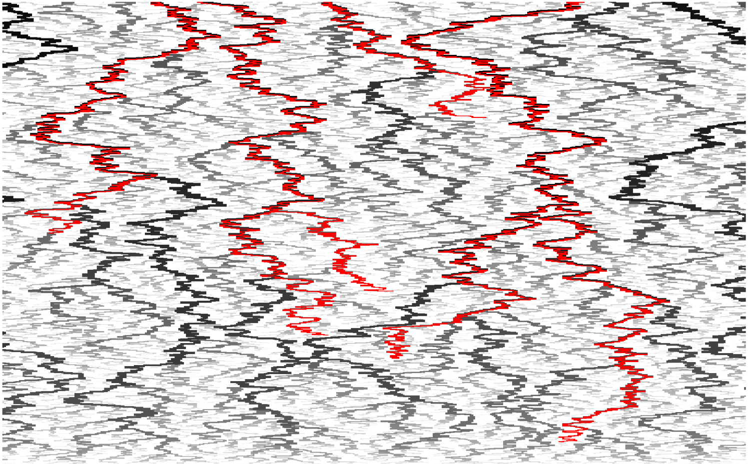

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Brownova síť

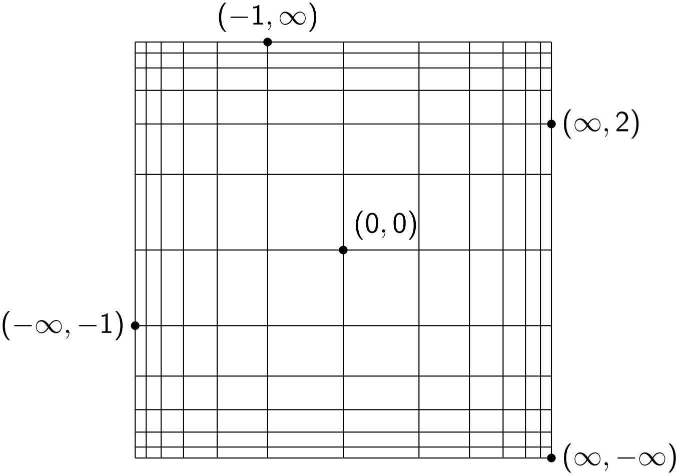
We want to give a rigorous construction of an object with the following informal description:

Coalescing Brownian motions,
started in each space-time point.

Moreover, we want to show that diffusively rescaled arrow configurations converge to such an object.

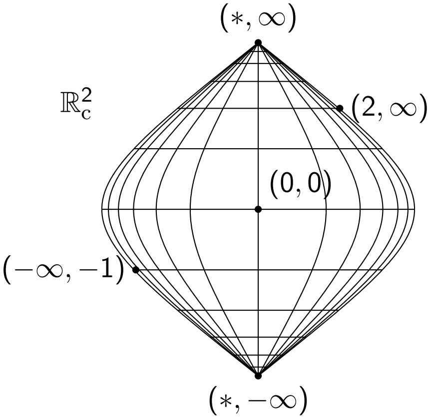
To this aim, we must first introduce the right topology.

Topological matters



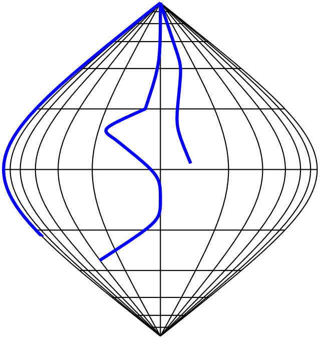
We compactify both space and time by adding points at $\pm\infty$.

Topological matters



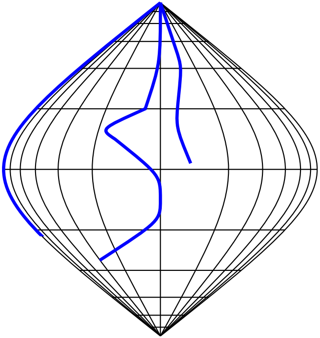
... and then contract $[-\infty, \infty] \times \{-\infty\}$ and $[-\infty, \infty] \times \{\infty\}$ to single points.

Topological matters



A *path* is a continuous function $\pi : [\sigma_\pi, \infty) \to [-\infty, \infty]$,
where $\sigma_\pi \in \mathbb{R}$ is the *starting time*.

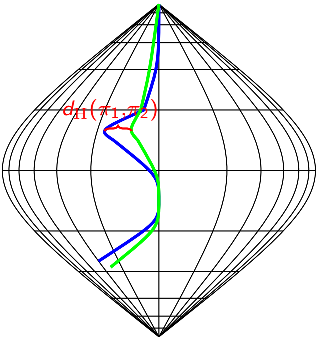
Topological matters



We identify a path with its graph

$$\{(\pi(t), t) : t \in [\sigma_\pi, \infty)\} \cup \{(*, \infty)\} \subset \mathbb{R}^2_c.$$

Topological matters

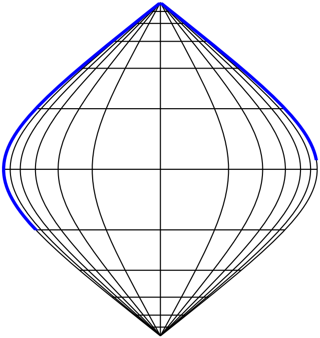


We equip the space Π of all paths with the *Hausdorff metric*

$$d_{\mathrm{H}}(\pi_1, \pi_2) := \sup_{z_1 \in \pi_1} \inf_{z_2 \in \pi_2} d(z_1, z_2) \vee \sup_{z_2 \in \pi_2} \inf_{z_1 \in \pi_1} d(z_1, z_2),$$

where d is a the metric on space-time $\mathbb{R}^2_{\mathrm{c}}$.

Topological matters



The set $\mathcal{U}$ consisting of all paths in an arrow configuration plus the trivial paths that are constantly $-\infty$ or $+\infty$ is a compact subset of the space of all paths Π.

Topological matters

We equip the space $\mathcal{K}(\Pi)$ of compact subsets of the path space Π with the Hausdorff metric:

$$d_{\mathrm{HH}}(\mathcal{U}_1,\mathcal{U}_2) := \sup_{\pi_1\in\mathcal{U}_1}\inf_{\pi_2\in\mathcal{U}_2} d_{\mathrm{H}}(\pi_1,\pi_2) \vee \sup_{\pi_2\in\mathcal{U}_2}\inf_{\pi_1\in\mathcal{U}_1} d_{\mathrm{H}}(\pi_1,\pi_2).$$

Here d_{H} is the metric on Π.

We let $\theta_\varepsilon : \mathbb{R}^2_{\mathrm{c}} \to \mathbb{R}^2_{\mathrm{c}}$ denote the diffusive scaling map

$$\theta_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t),$$

and set

$$\theta_\varepsilon(\pi) := \big\{\theta_\varepsilon(x,t) : (x,t)\in\pi\big\} \qquad \theta_\varepsilon(\mathcal{U}) := \big\{\theta_\varepsilon(\pi) : \pi\in\mathcal{U}\big\}.$$

Convergence to the Brownian web

Theorem [Fontes, Isopi, Newman, Ravishankar (2003)] The set $\mathcal{U}$ of paths in an arrow configuration (plus trivial paths) satisfies

$$\mathbb{P}[\theta_\varepsilon(\mathcal{U}) \in \cdot\,] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}[\mathcal{W} \in \cdot\,],$$

where $\Rightarrow$ denotes weak convergence of probability laws on $\mathcal{K}(\Pi)$ and $\mathcal{W}$ is a random compact set of paths called the *Brownian web*.

The Brownian web

The Brownian web is a the unique (in law) random compact subset of paths such that:

- In each deterministic point $z \in \mathbb{R}^2$ there almost surely starts precisely one path $p_z \in \mathcal{W}$.

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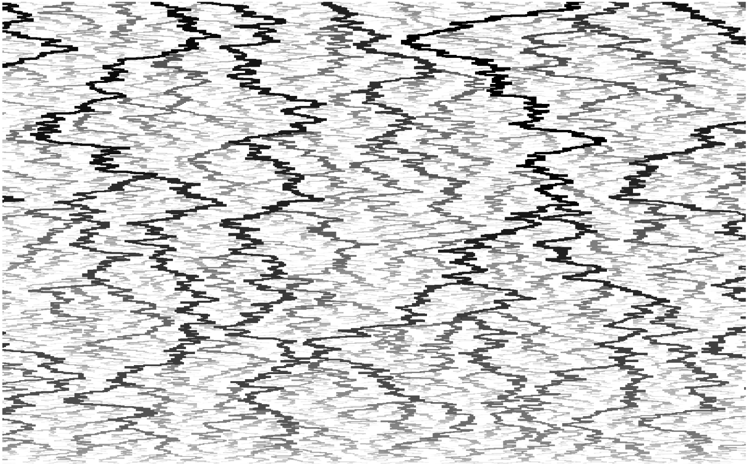
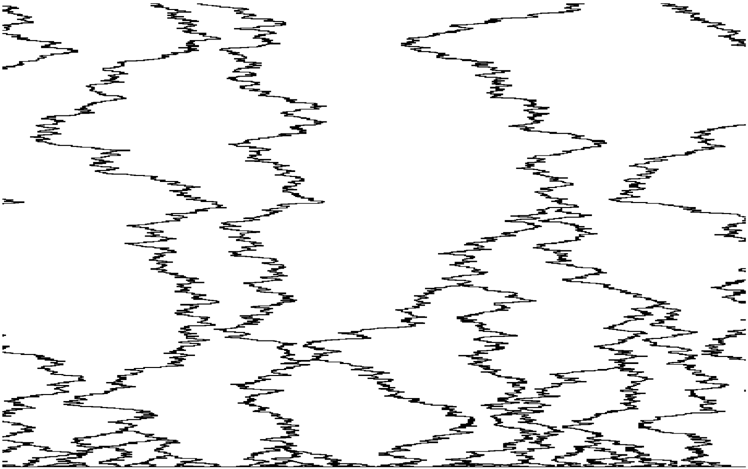
- The paths $p_{z_1}, \ldots, p_{z_k}$ starting in a finite collection $z_1, \ldots, z_k \in \mathbb{R}^2$ of deterministic points are distributed as coalescing Brownian motions.
- For each deterministic countable dense set $\mathcal{D} \subset \mathbb{R}^d$, the Brownian web $\mathcal{W}$ is the closure of the set $\mathcal{W}(\mathcal{D}) := \{p_z : z \in \mathcal{D}\}$ of paths starting in $\mathcal{D}$.

The Brownian web



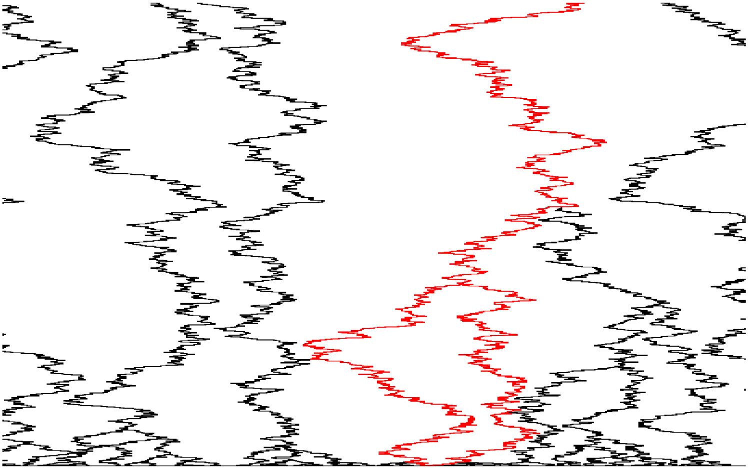
Artist's impression of the Brownian web.

The Brownian web



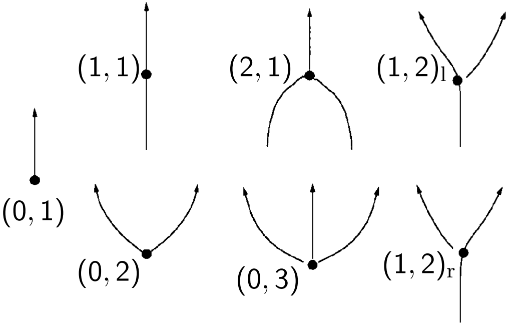
Paths starting at time zero.

The Brownian web



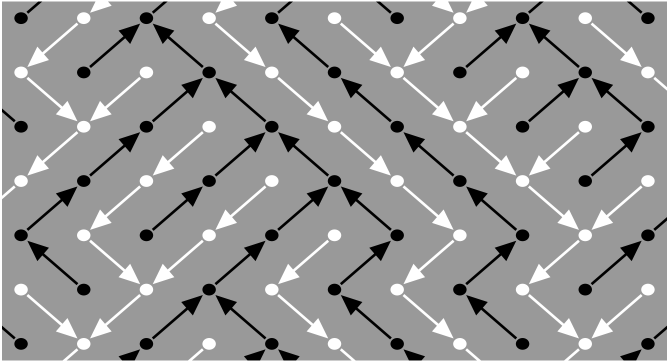
Even though at deterministic points z there a.s. starts a single path π_z, there exist random points that are the starting point of two paths.

Special points



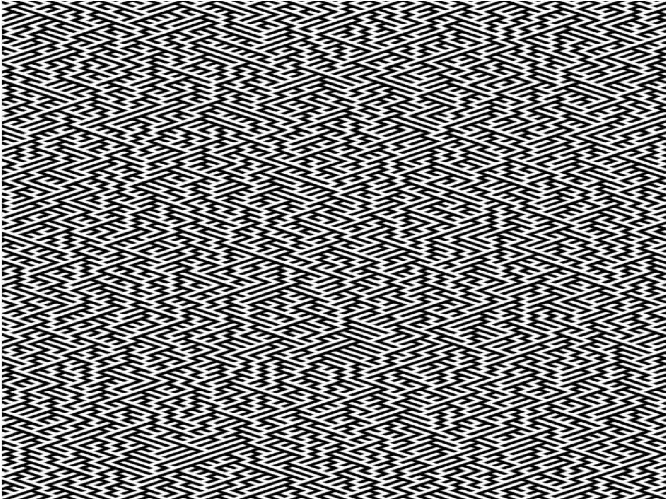
We can distinguish points in the plane according to the number of distinct paths entering and leaving a point.
In total, there are 7 types of points.

Dual arrows



Each arrow configuration defines a *dual* arrow configuration.

Dual Brownian web



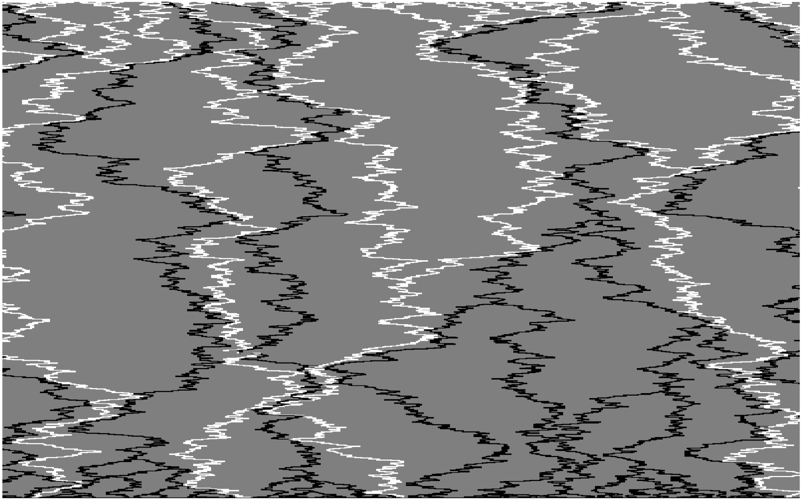
In the diffusive scaling limit, the dual arrow configuration converges to a dual Brownian web.

Dual Brownian web

Associated to each Brownian web $\mathcal{W}$, there is a *dual* Brownian web $\hat{\mathcal{W}}$ that is a.s. uniquely determined by $\mathcal{W}$ and equally distributed with $\mathcal{W}$ after a rotation over 180°.

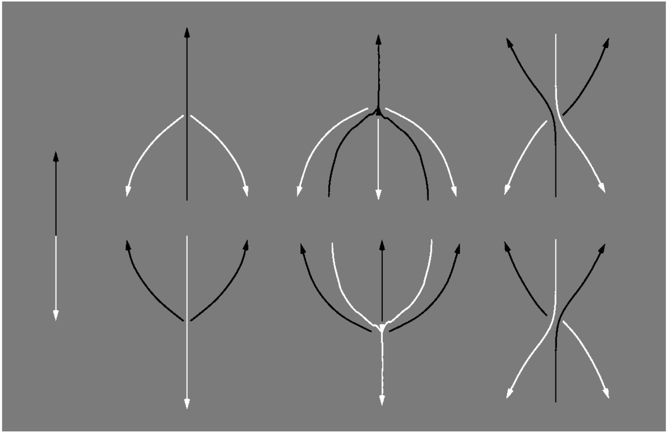
Dual paths reflect off forward paths with Skorohod reflection.

Dual Brownian web



Forward paths (black) and dual paths (white)
starting at two fixed times.

Special points revisited



The structure of the dual Brownian web at special points.

The voter model

Let $[0, 1]^{\mathbb{Z}}$ be the space of functions $x : \mathbb{Z} \to [0, 1]$.

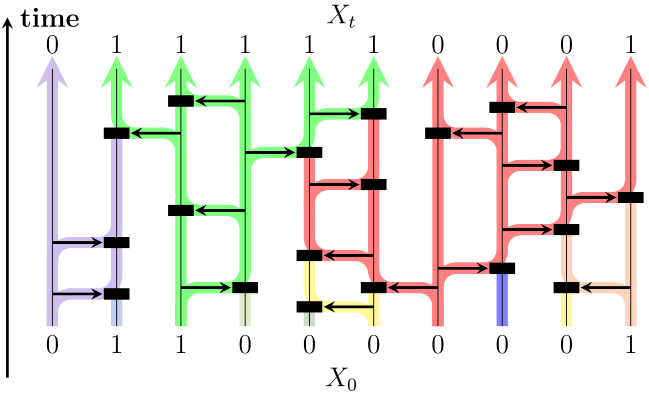
The infinite-type one-dimensional voter model is a Markov process $(X_t)_{t\geq 0}$ with state space $[0, 1]^{\mathbb{Z}}$.

We call $X_t(i) \in [0, 1]$ the *type* if site $i \in \mathbb{Z}$ at time $t \geq 0$.

Initially, $(X_0(i))_{i\in\mathbb{Z}}$ are i.i.d. uniformly distributed.

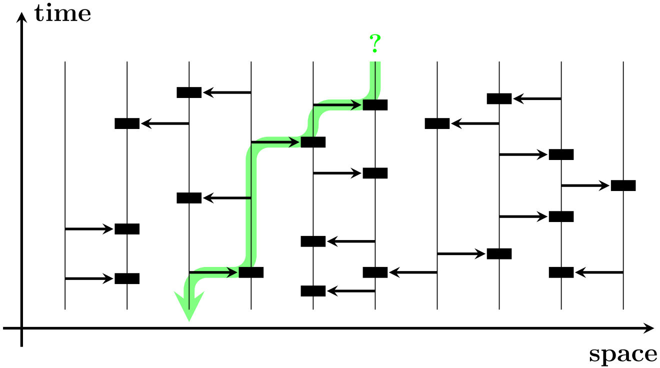
At times of a Poission point process with intensity one, the site i selects one of its neighbours at random and copies its type.

The voter model



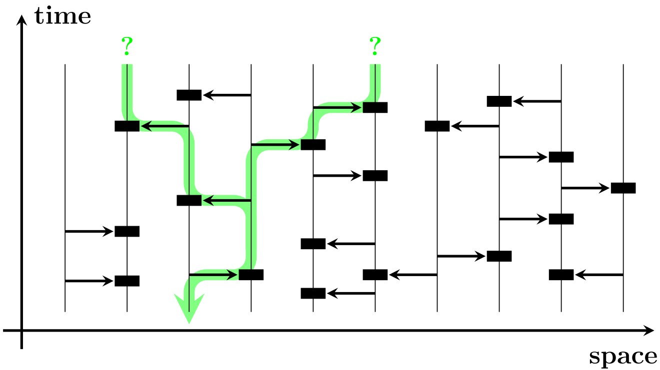
A one-dimensional voter model.

The voter model



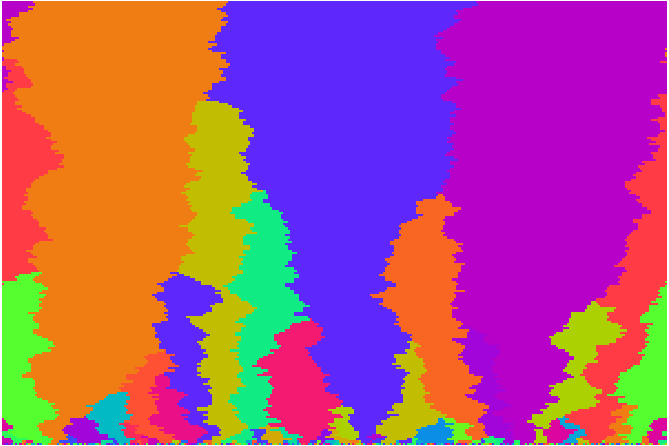
To find out the type of site i at time t,
we follow the arrows backwards.

The voter model



Ancestral lines started from different points coalesce when they meet.

The voter model



In the diffusive scaling limit, the ancestral lines converge to a dual Brownian web. Arratia (1979) initiated the study of the Brownian web with the aim of describing the scaling limit of the voter model.

The heat equation with Wright-Fisher noise

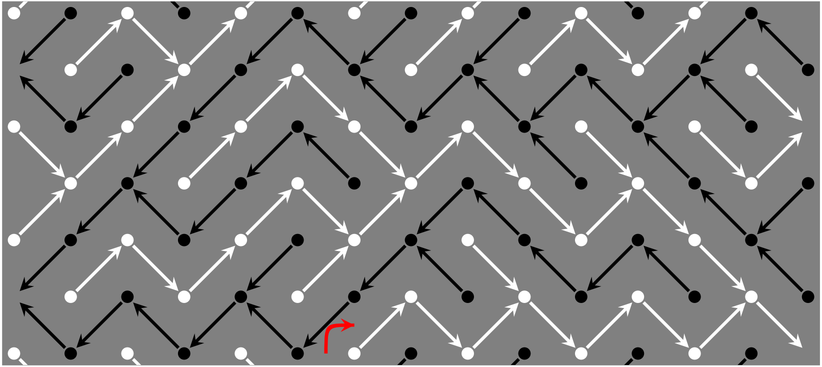
In 1988, T. Shiga studied the *heat equation with Wright-Fisher noise*

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(x,t) + \gamma\sqrt{u(x,t)\big(1-u(x,t)\big)}\,\partial W(x,t),$$

where $W(x,t)$ is space-time white noise. He showed that solutions to this PDE can via duality be expressed in terms of systems of Brownian motions that coalesce with rate γ times their intersection local time.

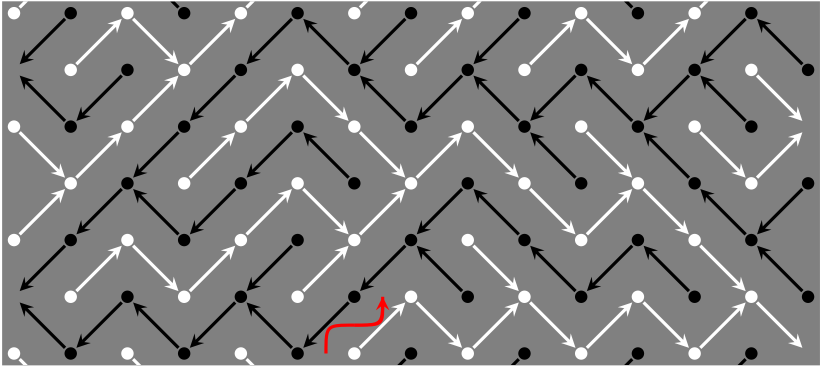
The continuum voter model with type space $\{0,1\}$ corresponds to the $\gamma \to \infty$ limit of this PDE.

The true self-repelling motion



In 1998, Bálint Tóth and Wendelin Werner used the Brownian web to describe the scaling limit of the *true self-repelling motion*.

The true self-repelling motion



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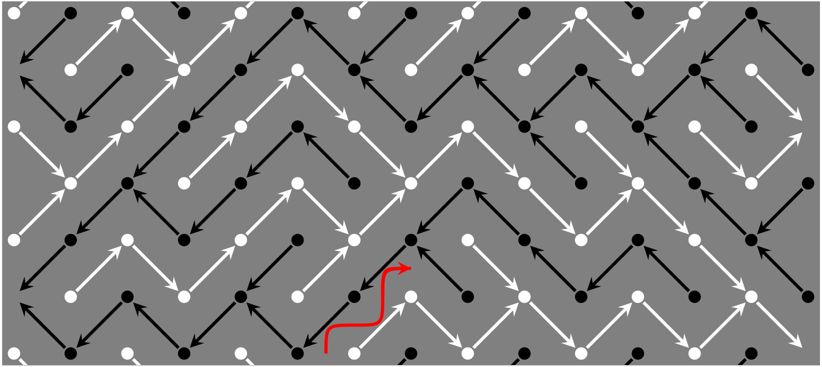

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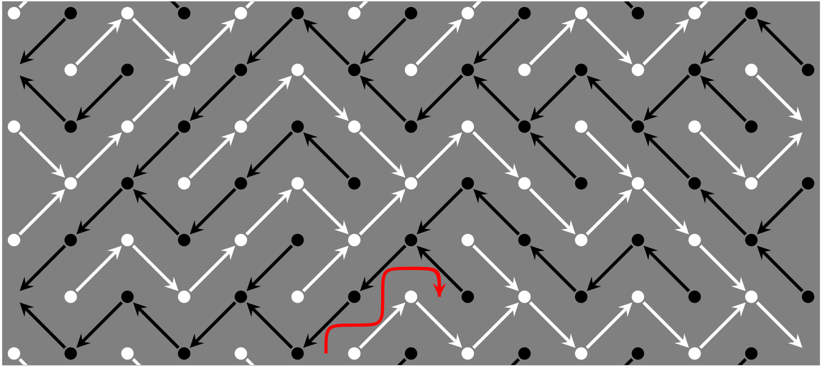

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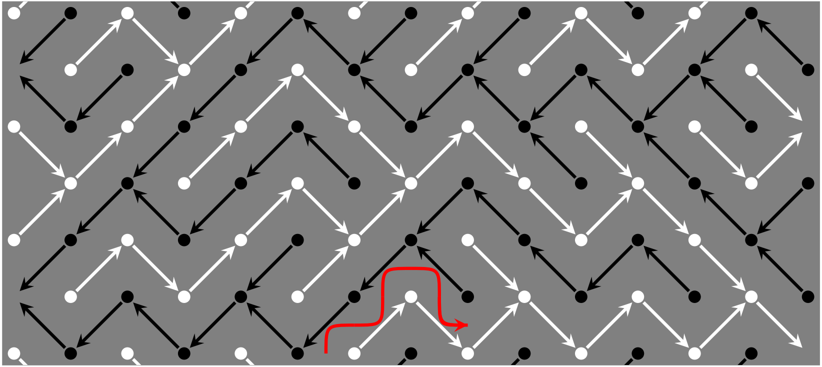

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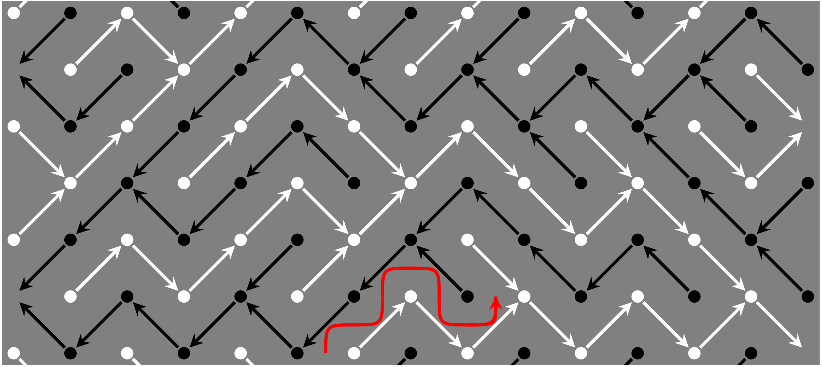

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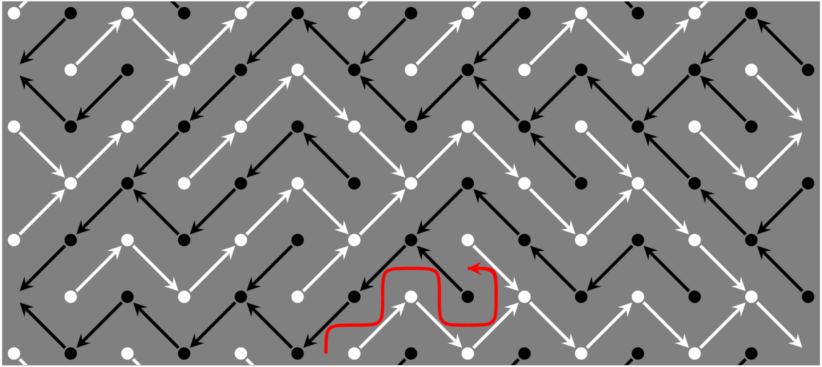

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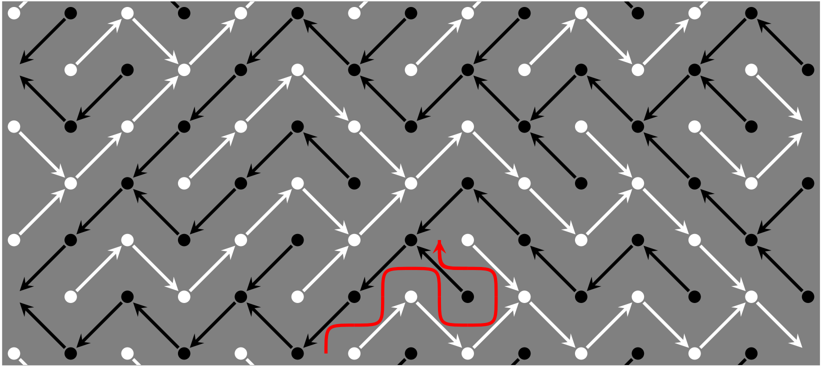

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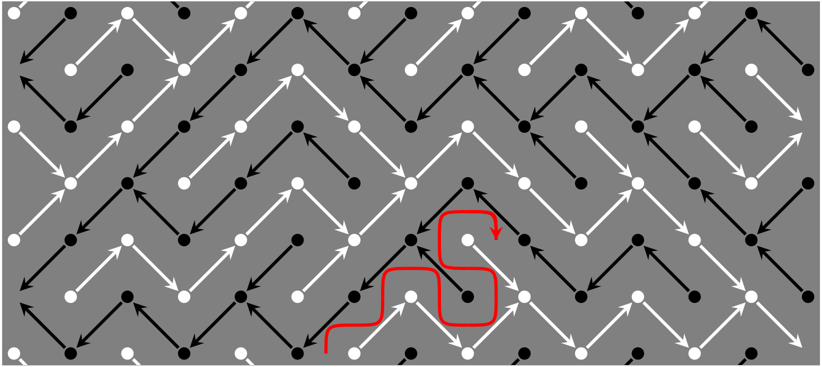

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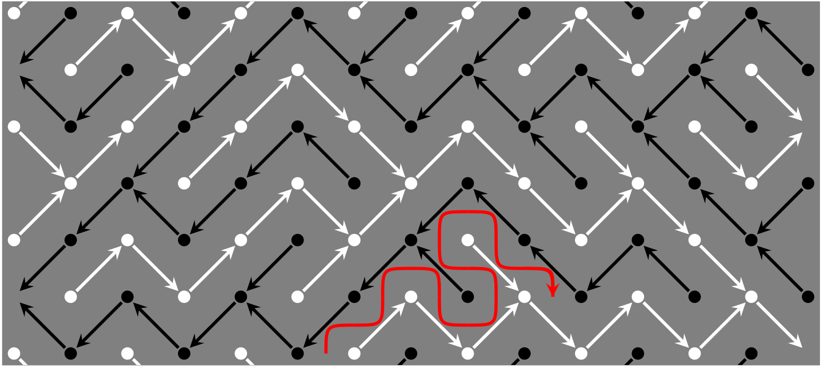

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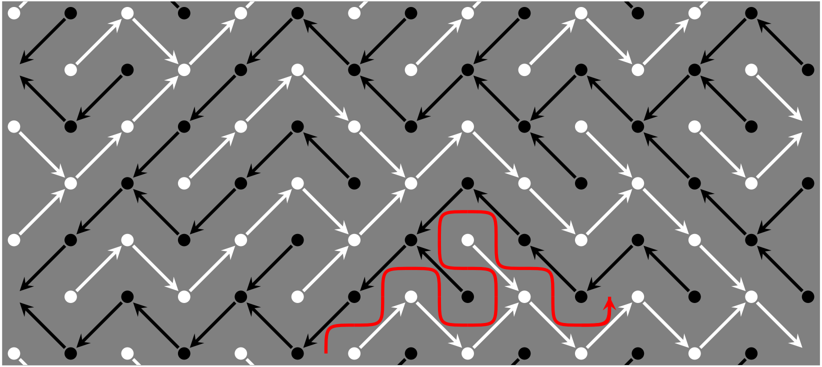

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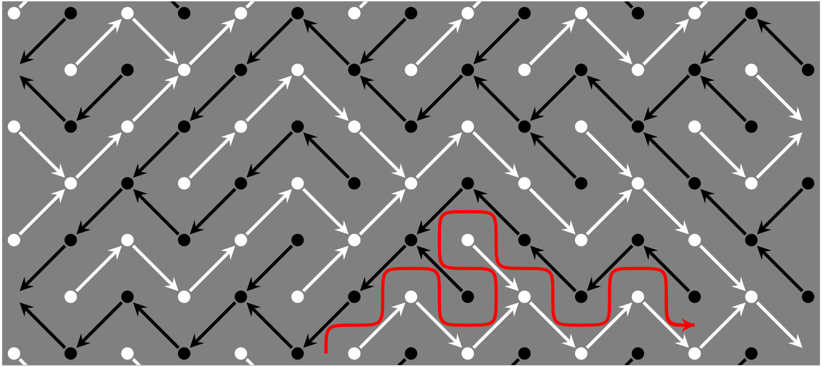

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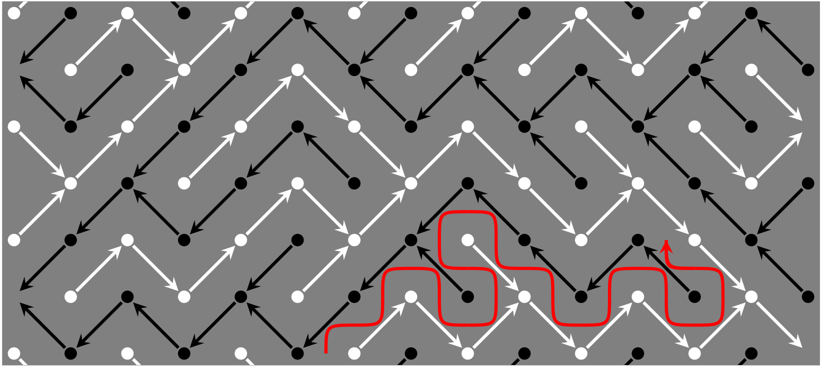

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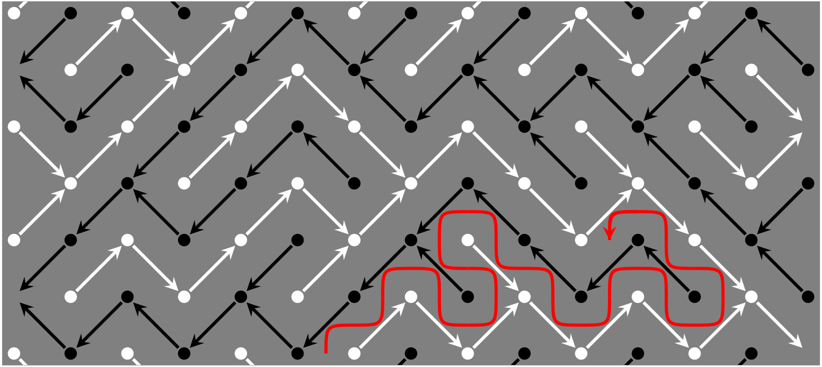

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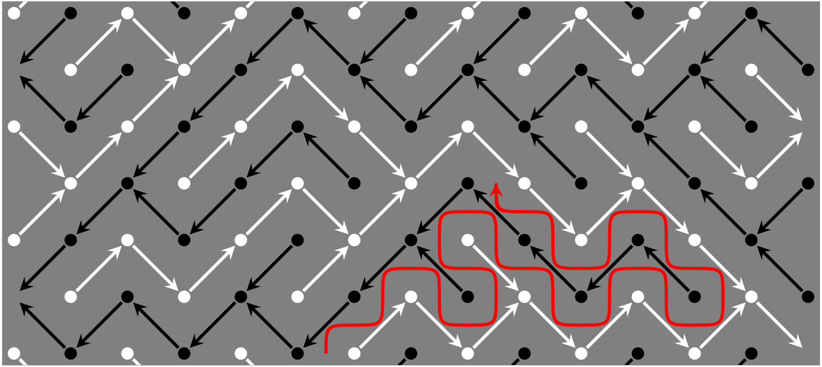

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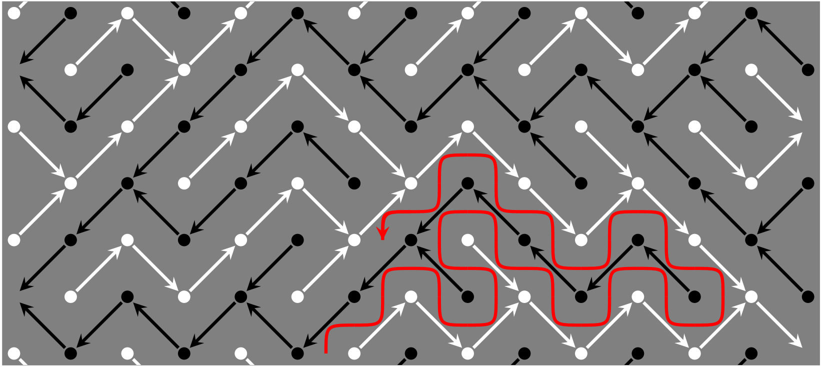

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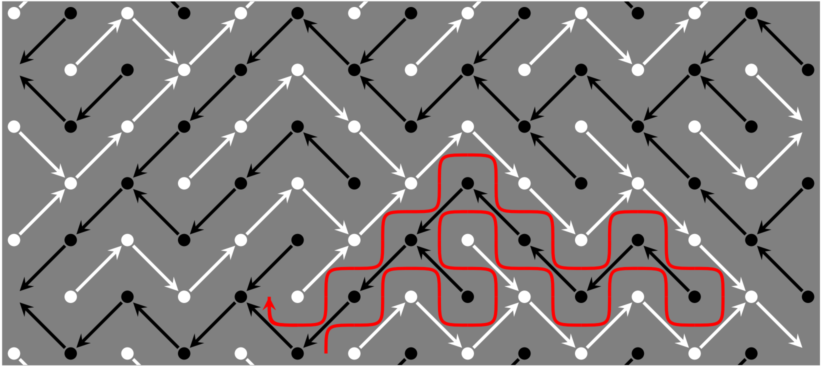

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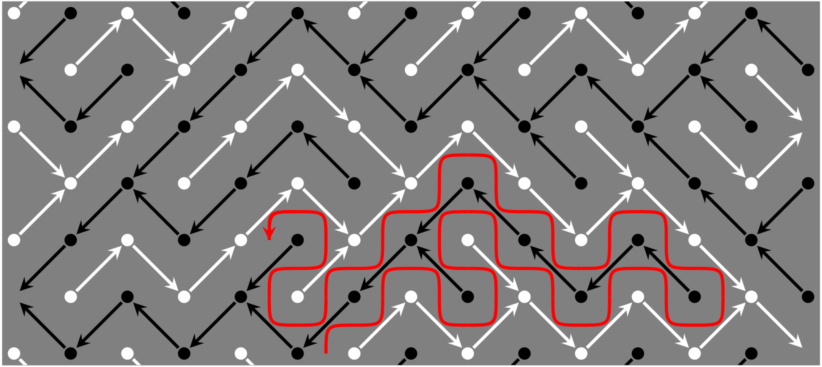

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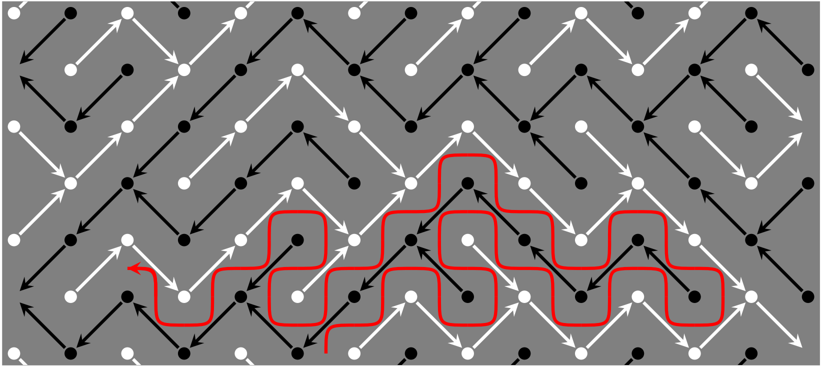

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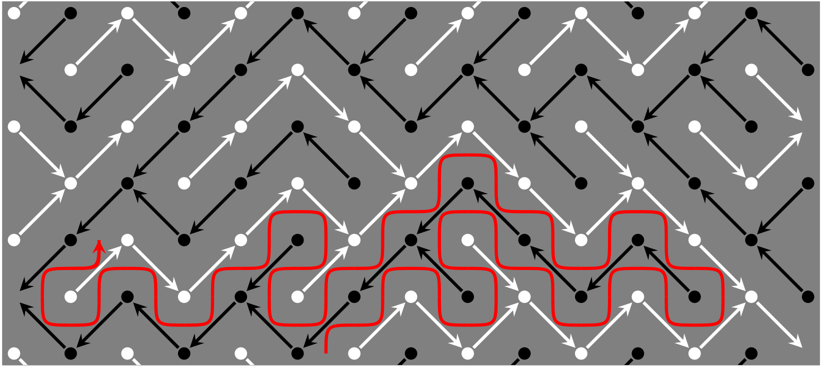

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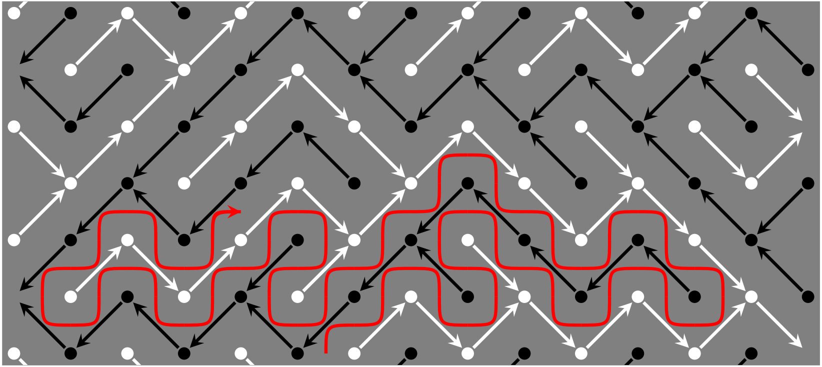

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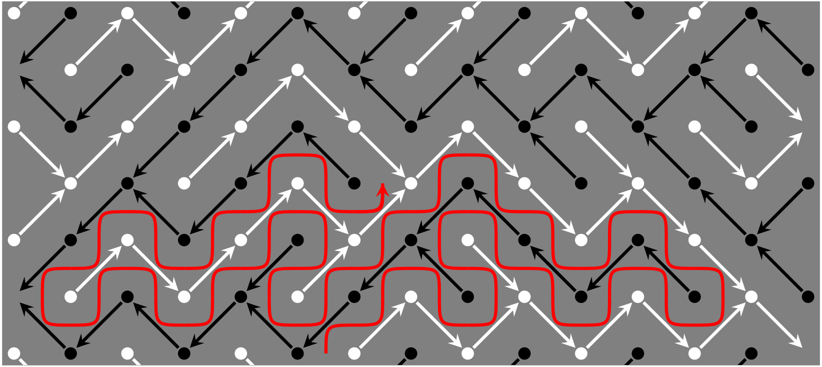

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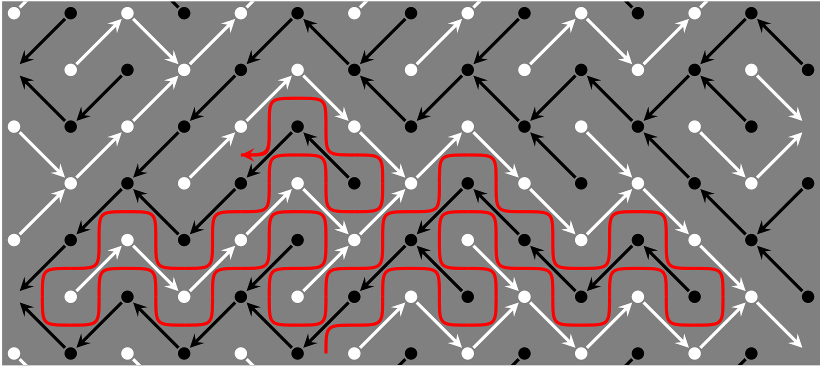

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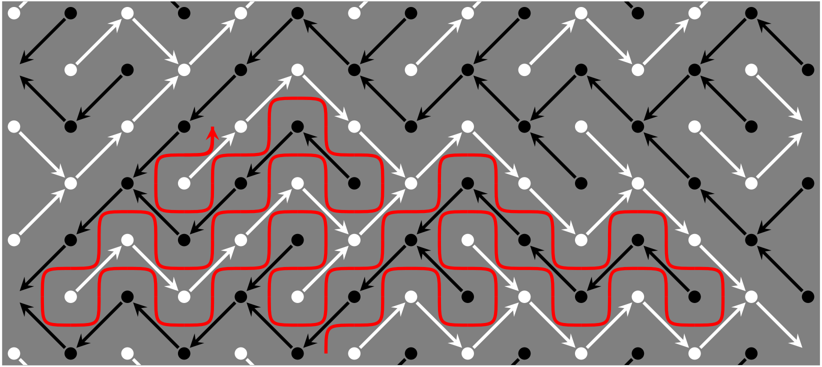

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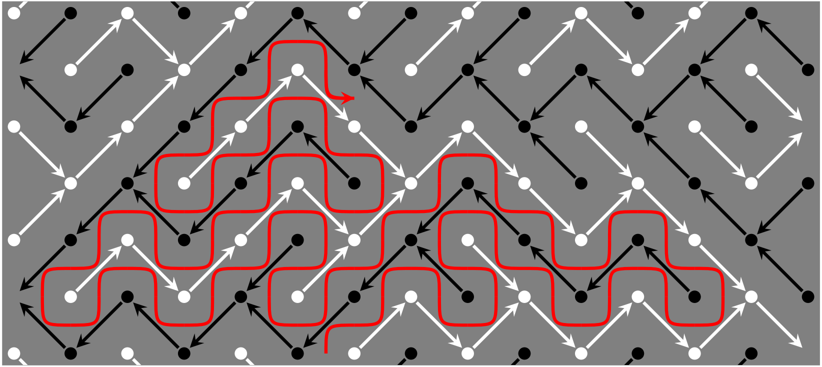

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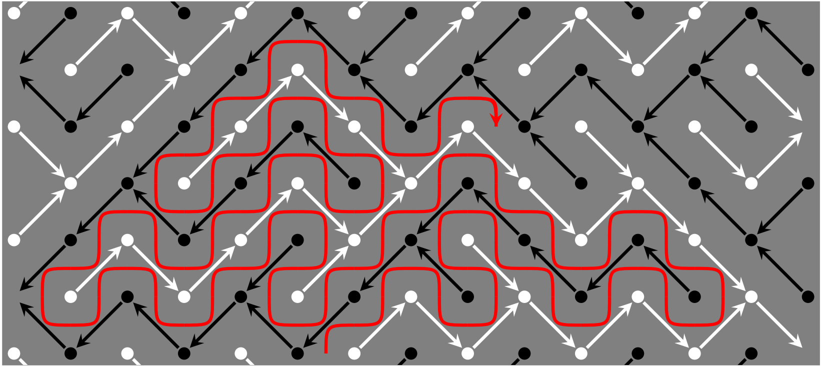

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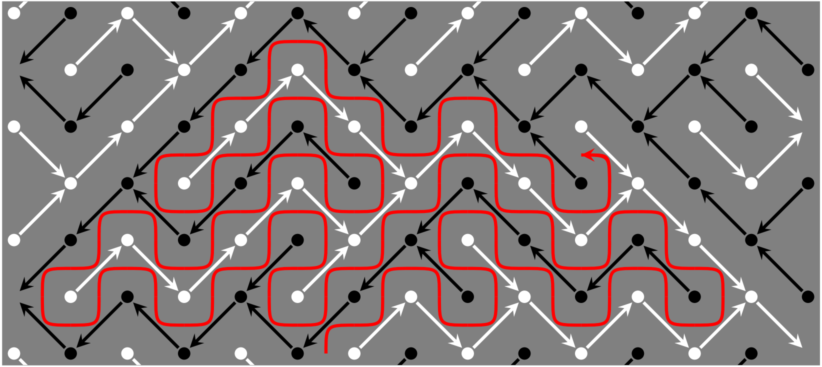

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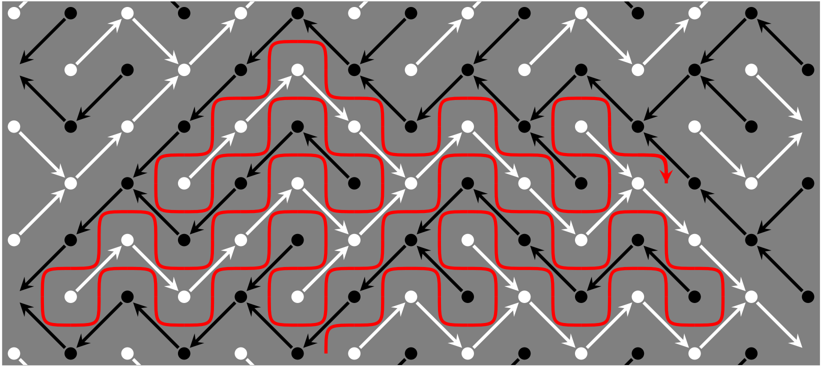

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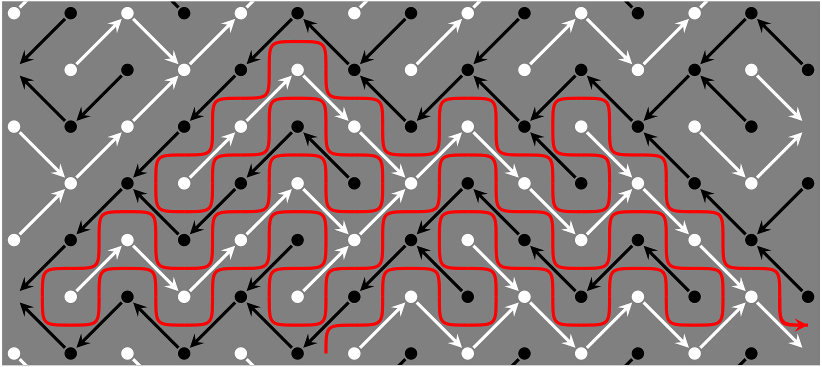

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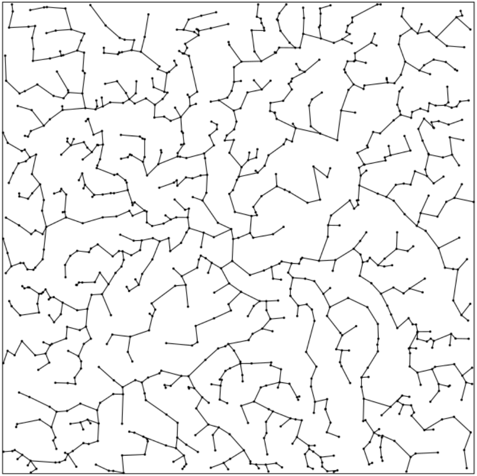
The directed spanning forest

In the *Euclidean directed spanning forest*[1] we place points in the plane according to a Poisson point process with intensity one, and we add one extra point at the origin.

We then connect each point to the nearest point that is closer to the origin.

[1] Thanks to Kateřina Pawlasová for the pictures.

The directed spanning forest



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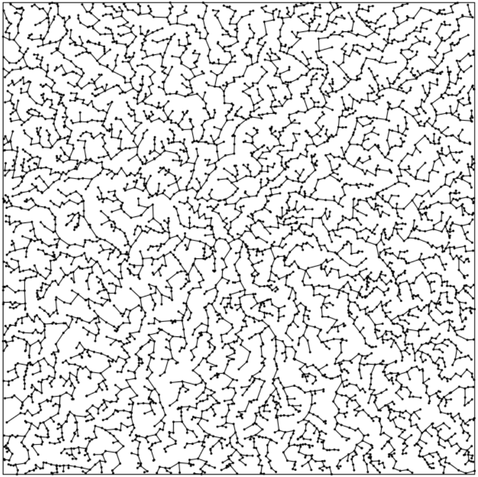

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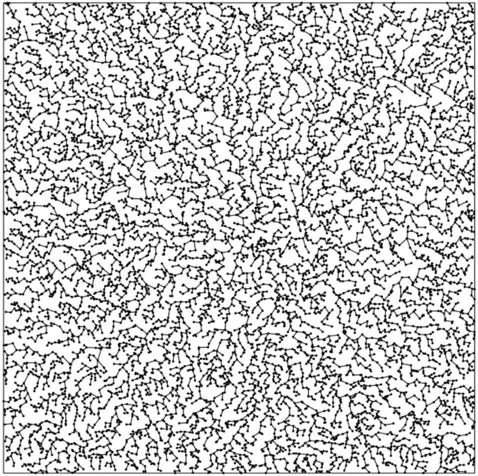


The directed spanning forest

In 2021, D. Coupier, K. Saha, A. Sarkar, and V.C. Tran showed that a directed version of this model scales to the Brownian web.

The 0-ballistic deposition model

In 2021, G. Cannizzaro and M. Hairer studied the *0-ballistic deposition model* that describes the growth of a random interface $H_t : \mathbb{Z} \to \mathbb{N}$.

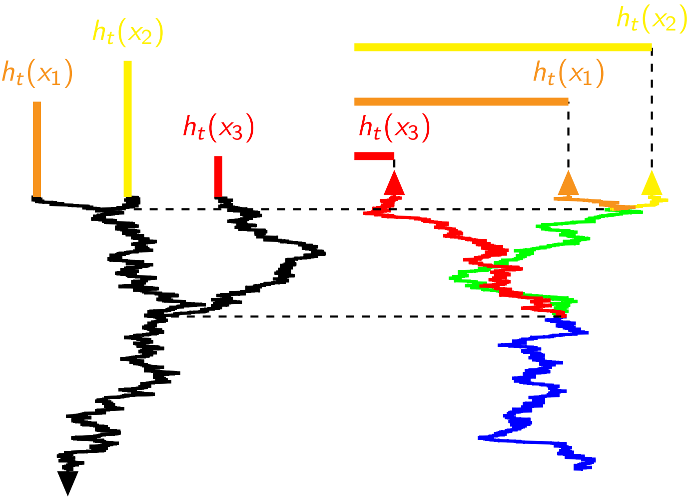
Each site $i \in \mathbb{Z}$ is updated with Poisson rate one, in such a way that:

$$H_t(i) \mapsto \begin{cases} H_t(i-1) & \text{with probability } 1/3, \\ H_t(i)+1 & \text{with probability } 1/3, \\ H_t(i+1) & \text{with probability } 1/3. \end{cases}$$

The 0-ballistic deposition model

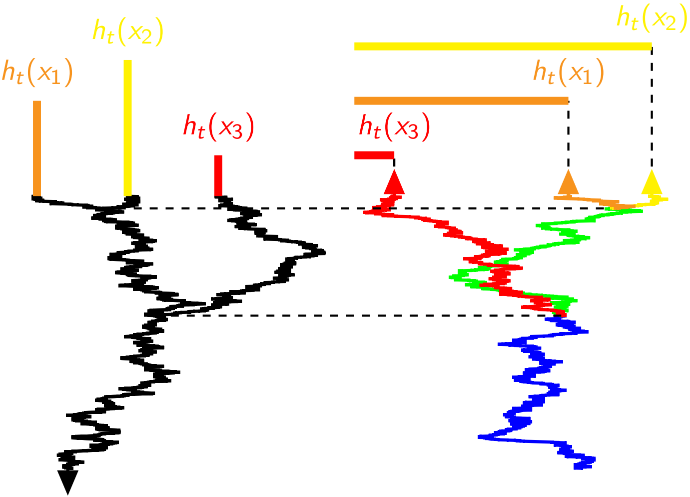
After subtracting the mean, we are interested in the diffusive scaling limit of the interface.

The 0-ballistic deposition model



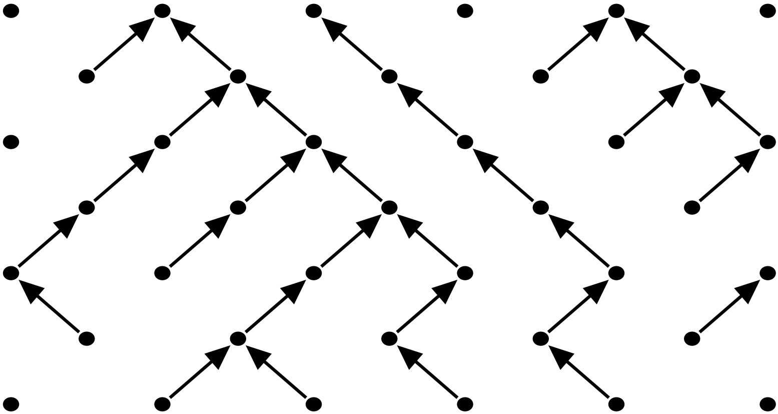
To determine the heights $h_t(x_i)$ in points $x_1, \ldots, x_n$, we first construct downward coalescing Brownian motions from (x_i, t).

The 0-ballistic deposition model



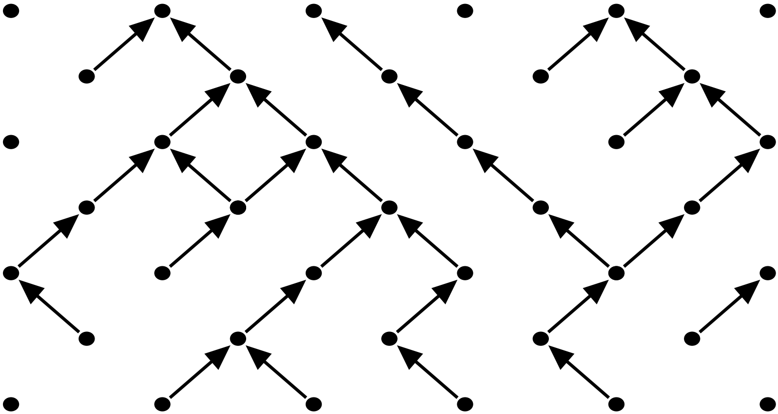
The heights are then determined by upward Brownian motions that branch with the tree structure of the downward Brownian motions.

Branching and coalescence



We can change an arrow configuration by drawing with probability ε two arrows, one to the left and one to the right.

Branching and coalescence



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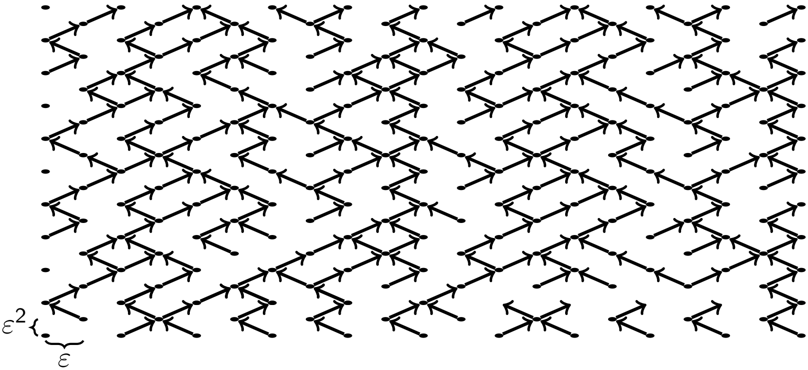
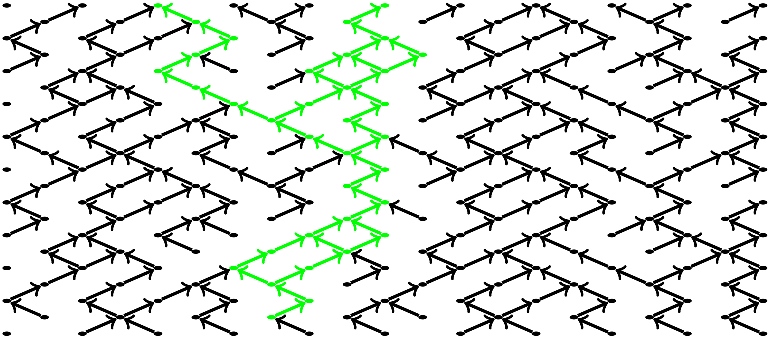


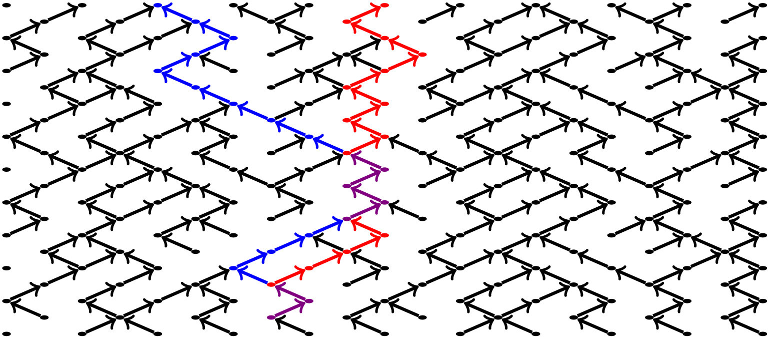
We again rescale space by ε and time by ε^2.

Branching and coalescence



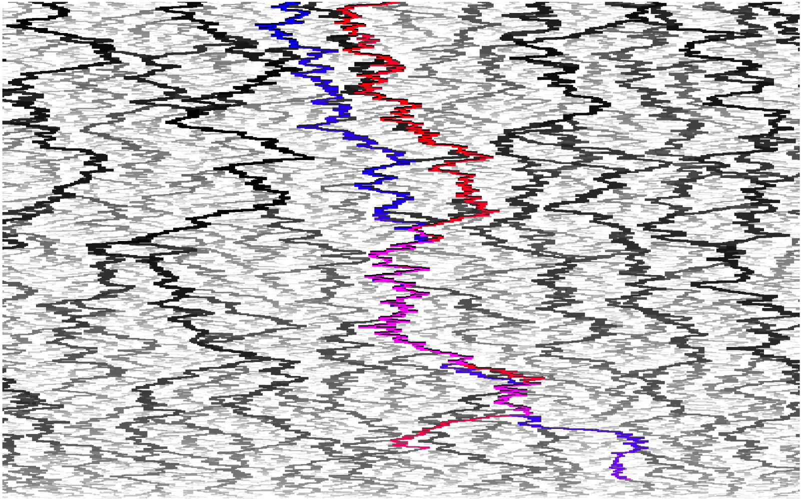
In each space-time point, there now start many different paths.

Left and right paths



At each point, there start a unique left-most and right-most path.

Left and right paths



After diffusive rescling, these converge
to drifted Brownian motions.

Left and right paths

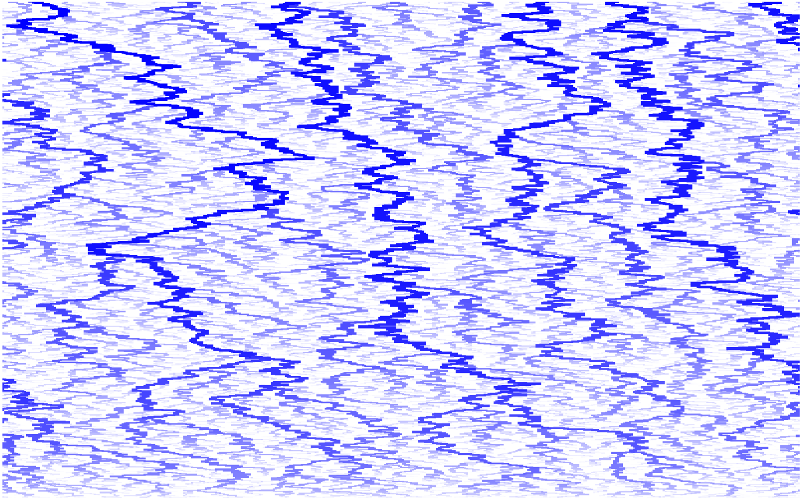
The joint law of left and right paths is described by the stochastic differential equation:

$$\mathrm{d}L_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t - \mathrm{d}t,$$
$$\mathrm{d}R_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \mathrm{d}t,$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t satisfy $L_t \leq R_t$ for all $t \geq \tau := \inf\{u \geq 0 : L_u = R_u\}$.

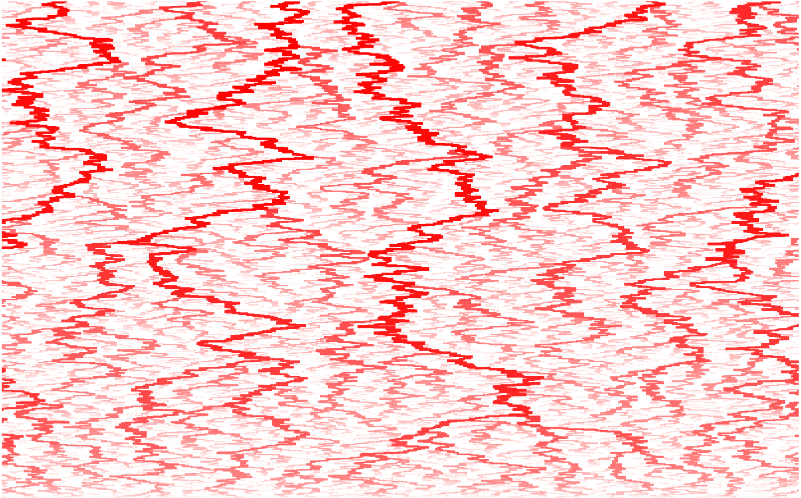
The set $\{t : L_t = R_s\}$ is nowhere dense and has positive Lebesgue measure.

Left Brownian web



All left-most paths form a left Brownian web. . .

Right Brownian web



. . . and the right-most paths form a right Brownian web.

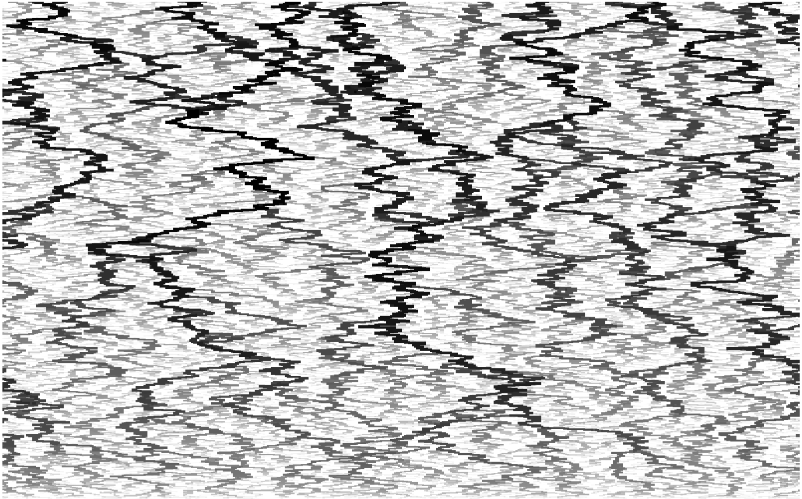
Convergence to the Brownian net

Theorem [Sun, S. (2008)] Let $\mathcal{U}_\varepsilon$ be the collection of paths in an arrow configuration with branching probability ε. Then

$$\mathbb{P}[\theta_\varepsilon(\mathcal{U}_\varepsilon) \in \cdot\,] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}[\mathcal{N} \in \cdot\,],$$

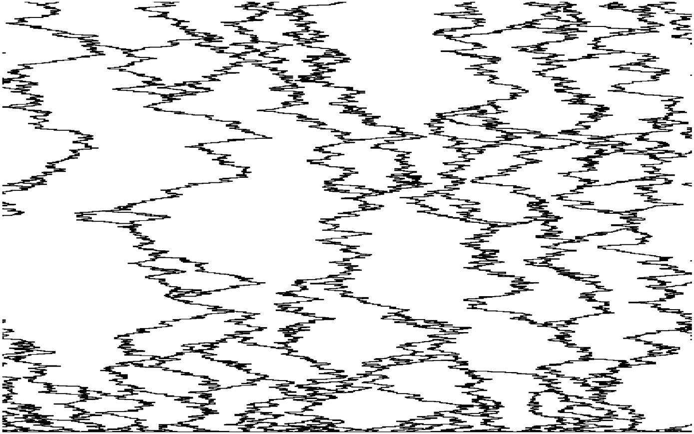
where $\Rightarrow$ denotes weak convergence on the space $\mathcal{K}(\Pi)$ of compact sets of paths, and the limiting object $\mathcal{N}$ is called the *Brownian net*.

Brownian net



Artist's impression of the Brownian net.

Brownian net



The collection of paths starting at time zero.

The branching-coalescing point set

For each closed set $A \subset \mathbb{R}$,

$$\xi_t := \{\pi_t : \exists \pi \in \mathcal{N} \text{ s.t. } \sigma_\pi = 0,\ \pi_0 \in A\} \qquad (t \geq 0)$$

defines a Markov process $(\xi_t)_{t \geq 0}$ with values in the space of closed subsets of $\mathbb{R}$.

(i) Invariant law: the law of a Poisson point process with intensity 1.

(ii) For deterministic $t > 0$, the set ξ_t is a.s. a locally finite subset of $\mathbb{R}$.

(iii) There exists a dense set of random times at which ξ_t has no isolated points.

Pott's model

A one-dimensional Potts model with Glauber dynamics is a Markov process $(X_t)_{t\geq 0}$ taking values in the space $\{1,\ldots,q\}^{\mathbb{Z}}$ of functions $x:\mathbb{Z}\to\{1,\ldots,q\}$.

For $x\in\{1,\ldots,q\}^{\mathbb{Z}}$, let

$$N_i^x(\sigma):=\sum_{j\in\{i-1,i+1\}} 1_{\{x(j)=\sigma\}}$$

denote the number of neighbours of $i\in\mathbb{Z}$ that have the value $\sigma\in\{1,\ldots,q\}$.

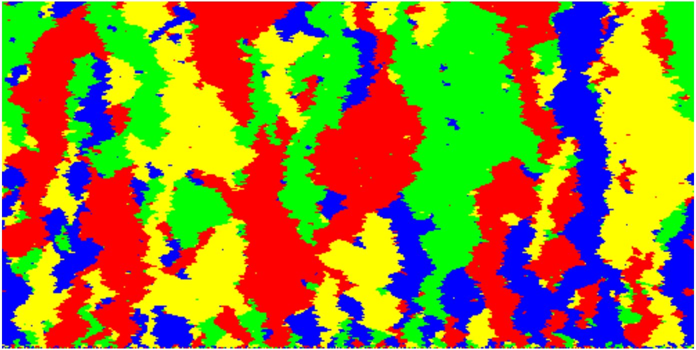
Each site i is updated with Poisson rate one and chooses a new value according to the law

$$\mu_i^x(\sigma):=\frac{1}{Z_i^x}e^{\beta N_i^x(\sigma)}\qquad\big(\sigma\in\{1,\ldots,q\}\big),$$

where Z_i^x is a normalisation constant.

We are interested in the low temperature limit $\beta\to\infty$.

Pott's model



A low temperature one-dimensional Potts model
with Glauber dynamics.

Voter model perturbations

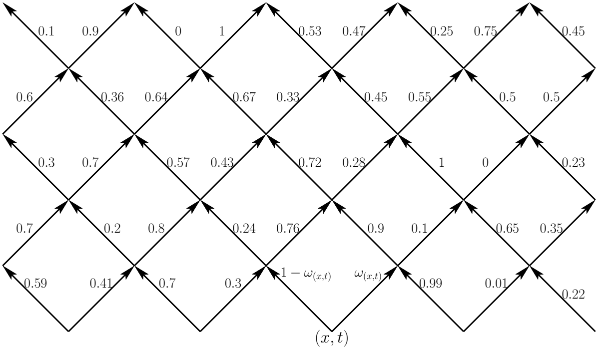
C.M. Newman, K. Ravishankar, and E. Schertzer (2015) studied coalescing random walks that branch with probability ε and die with probability ε^2.

The diffusive scaling limit is called the *Brownian net with killing.*

In 2017, they used the Brownian net with killing to describe the scaling limit of low temperature one-dimensional Potts models with Glauber dynamics.

More generally, their method applies to a wide class of voter model perturbations.

Random space-time environment



Fix a probability law μ on $[0,1]$.
Let $(\omega_z)_{z\in\mathbb{Z}^2_{\rm even}}$ be i.i.d. $[0,1]$-valued r.v.'s with law μ.

A measure-valued process

Fix some probability measure ρ_0 on $\mathbb{Z}_{\rm even}$, and define inductively, for $(x,t) \in \mathbb{Z}^2_{\rm even}$:

$$\rho_t(x) := \omega_{(x-1,t-1)}\rho_{t-1}(x-1) + (1-\omega_{(x+1,t-1)})\rho_{t-1}(x+1).$$

Interpretation: in the time step from t to $t+1$, a $\omega_{(x,t)}$ fraction of the mass at x is sent to $x+1$ and the rest is sent to $x-1$.

Then $(\rho_t)_{t\geq 0}$ is a Markov chain taking values alternatively in the probability measures on $\mathbb{Z}_{\rm even}$ and $\mathbb{Z}_{\rm odd}$.

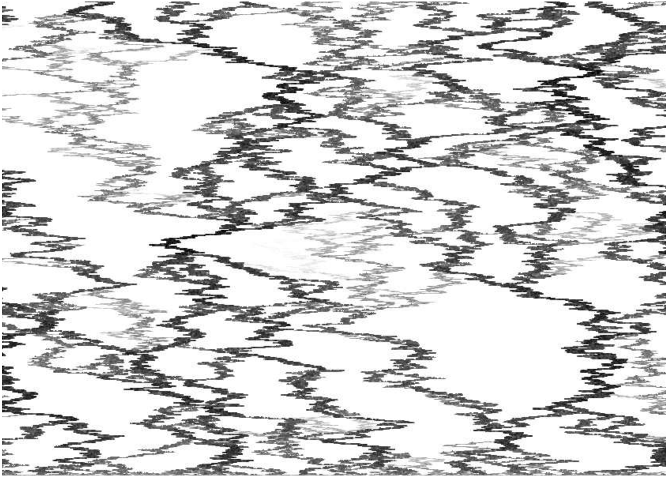
Theorem [Le Jan & Raimond '04, Howitt & Warren '06]
Let $\varepsilon_n \to 0$ and let $\rho_t^{(n)}(x)$ be Markov chains defined by splitting laws μ_n satisfying:

$$\begin{array}{ll}\text{(i)} & \varepsilon_n^{-1} \int 2(q - \tfrac{1}{2})\mu_n(\mathrm{d}q) \underset{n\to\infty}{\longrightarrow} \beta, \\ \text{(ii)} & \varepsilon_n^{-1} q(1-q)\mu_n(\mathrm{d}q) \underset{n\to\infty}{\Longrightarrow} \nu(\mathrm{d}q),\end{array}$$

with $\beta \in \mathbb{R}$ and ν a finite measure on $[0,1]$.

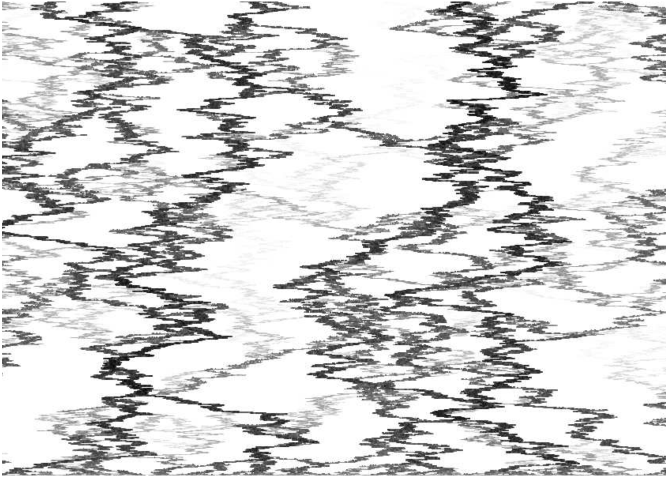
Rescale diffusively: $\tilde{\rho}^{(n)}_{\varepsilon_n^2 t}(\varepsilon_n x) := \rho_t^{(n)}(x)$. Then $\tilde{\rho}^{(n)} \Rightarrow \rho$, where $(\rho_t)_{t\geq 0}$ is a Markov process taking values in the probability measures on $\mathbb{R}$, with dynamics characterized by β and ν.

Howitt-Warren flows



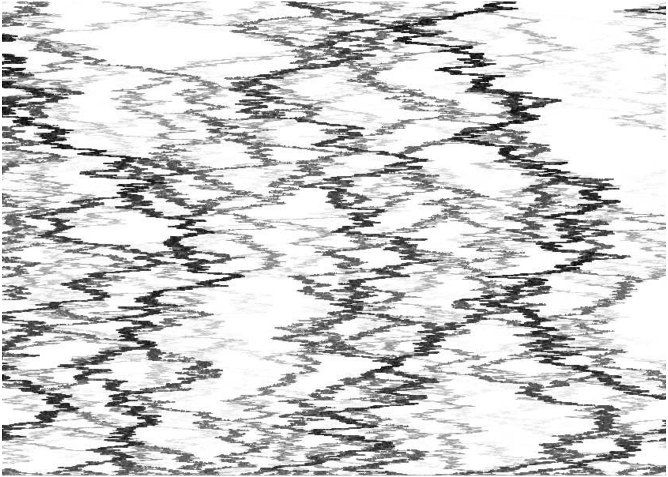
The equal splitting flow: $\beta = 0$ and $\nu = \delta_{1/2}$.

Howitt-Warren flows



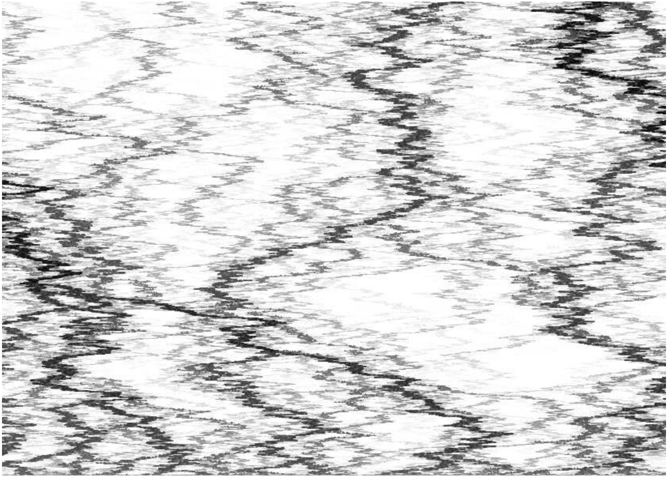
The process with $\beta = 0$ and $\nu(\mathrm{d}q) = 6q(1-q)\mathrm{d}q$.

Howitt-Warren flows



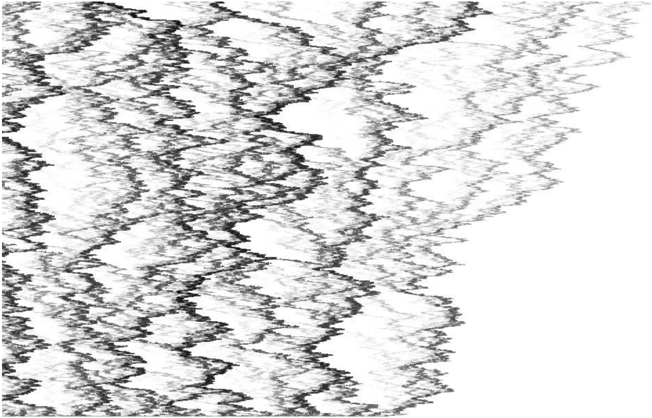
Le Jan-Raimond flow: $\beta = 0$ and $\nu(\mathrm{d}q) = \mathrm{d}q$ (reversible!).

Howitt-Warren flows



The erosion flow: $\beta = 0$ and $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Howitt-Warren flows



One-sided erosion flow: $\beta = 0$ and $\nu = \delta_1$.

Howitt-Warren flows

In 2010, E. Schertzer, R. Sun & J.S. showed that Howitt-Warren flows can be constructed with the help of the Brownian web.

If the speeds

$$\beta_+ := \beta + 2\int q^{-1}\nu(\mathrm{d}q),$$
$$\beta_- := \beta - 2\int (1-q)^{-1}\nu(\mathrm{d}q),$$

are finite, then they are supported on a Brownian let with left speed β_- and right speed β_+.