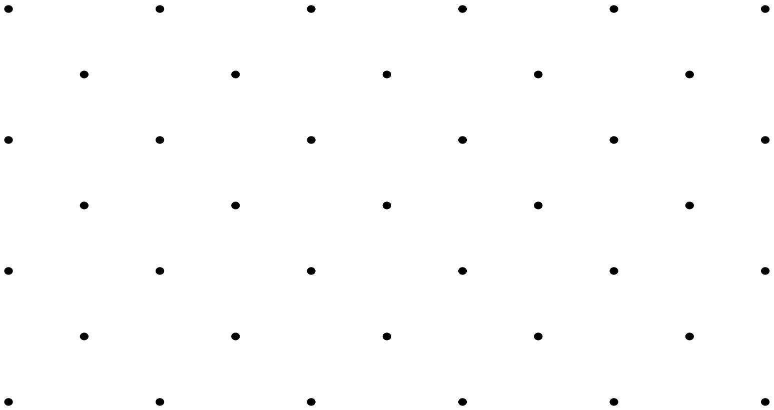
The Brownian net and its meshes

Jan M. Swart

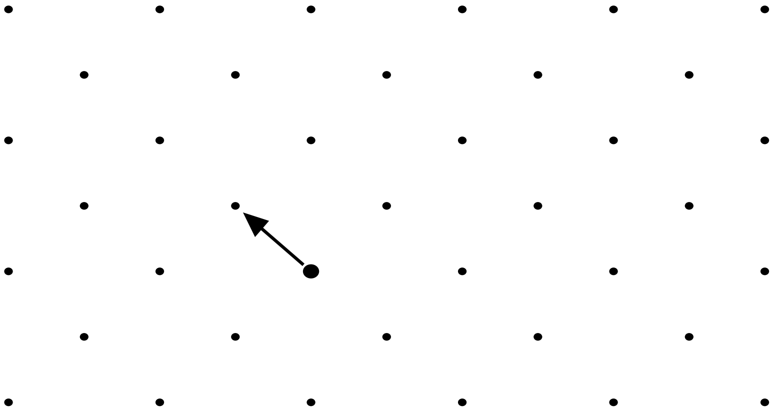
March 31, 2025

A discrete net



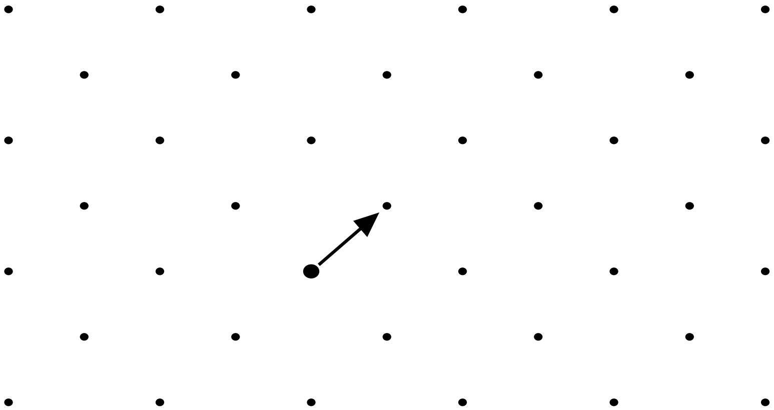
$$\mathbb{Z}^2_{\text{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}.$$

A discrete net



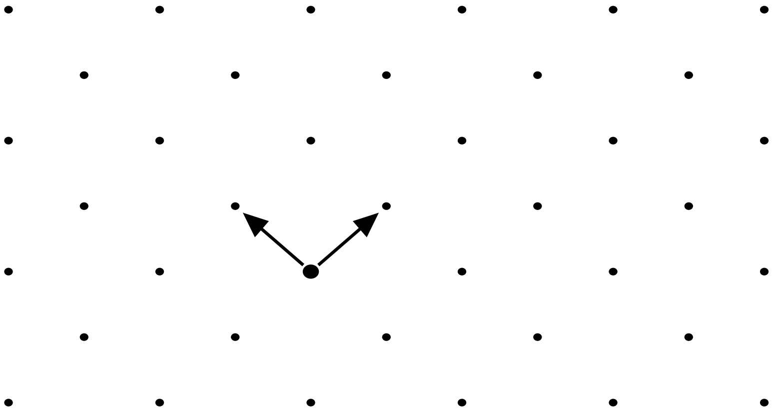
With probability $\frac{1}{2}(1-\varepsilon)$ we draw an arrow to the left.

A discrete net



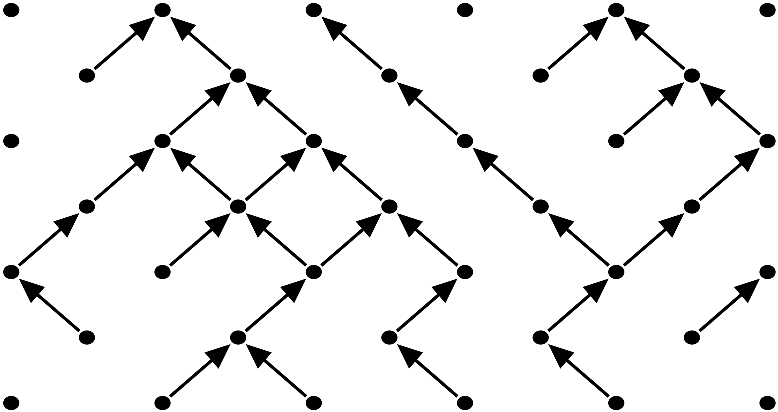
With probability $\frac{1}{2}(1-\varepsilon)$ we draw an arrow to the right.

A discrete net



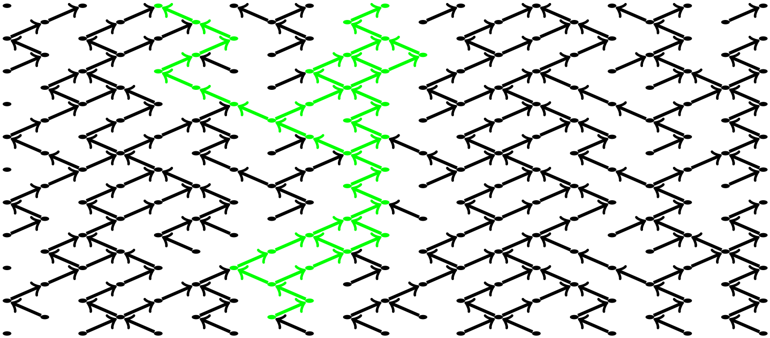
With probability ε we draw both arrows.

A discrete net



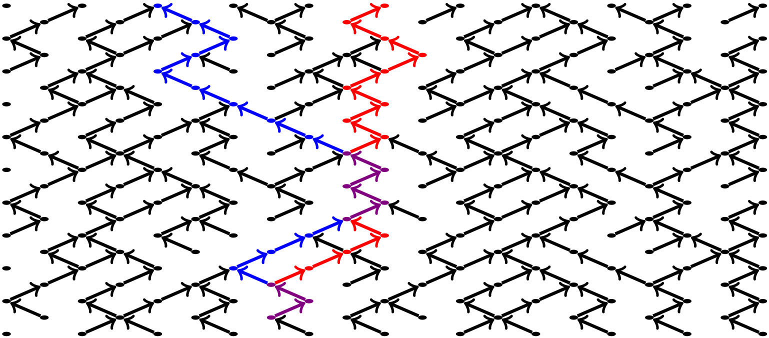
Independently for each point.

A discrete net



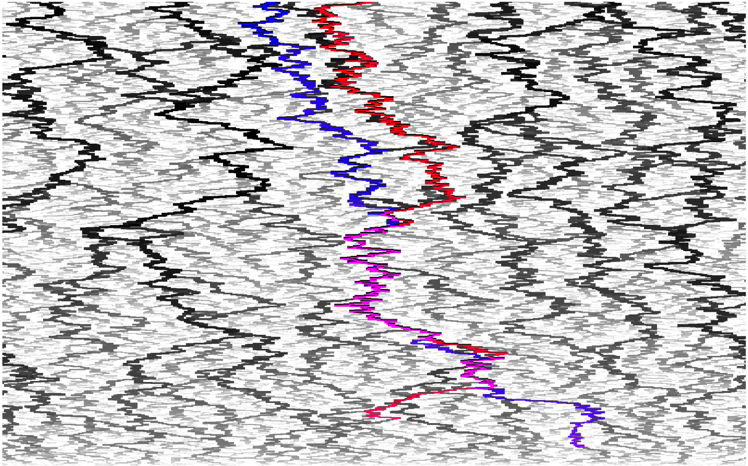
In each point there start many paths.

A discrete net



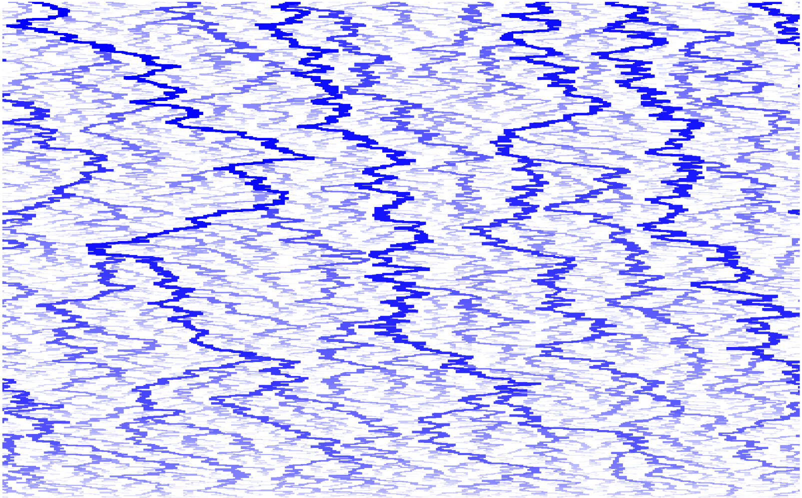
But in each point there starts a unique left and right path.

The left-right Brownian web



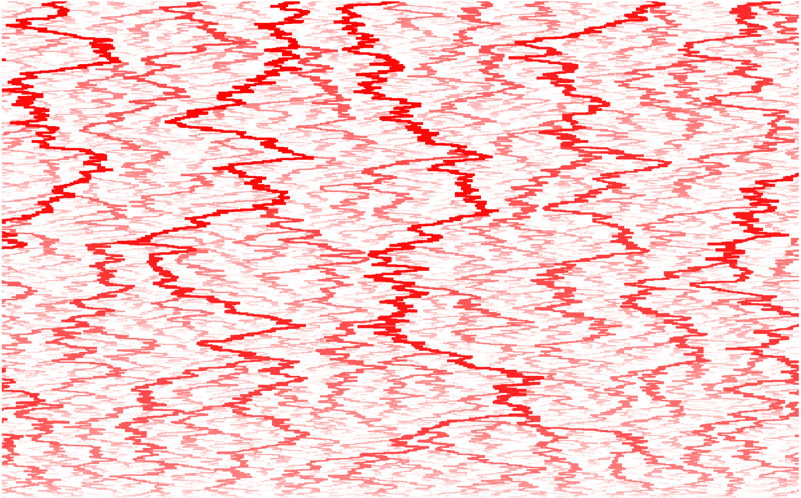
After scaling space by ε and time by ε^2, left and right paths converge to Brownian motions with drift ± 1.

The left-right Brownian web



The left paths converge to a left Brownian web

The left-right Brownian web



and the right paths converge to a right Brownian web.

The left-right Brownian web

The interaction between left and right paths is described by the SDE

$$\mathrm{d}L_t = 1_{\{L_t \neq R_t\}}\mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}}\mathrm{d}B^{\mathrm{s}}_t - \mathrm{d}t,$$

$$\mathrm{d}R_t = 1_{\{L_t \neq R_t\}}\mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}}\mathrm{d}B^{\mathrm{s}}_t + \mathrm{d}t,$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t satisfy the constraint $L_t \leq R_t$ for each $t \geq \tau := \inf\{u \geq 0 : L_u = R_u\}$.

The set $\{t : L_t = R_s\}$ is nowhere dense.
and has positive Lebesgue measure.

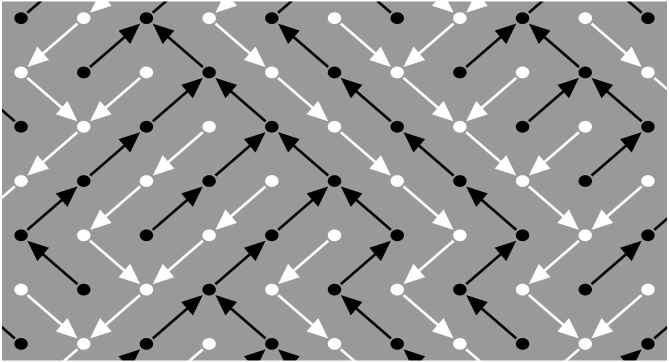
The Brownian web

Let $\Pi^{\uparrow}$ be the set of upward paths, equipped with the Brownian web topology. There exists a random compact set $\mathcal{W} \subset \mathcal{K}(\Pi^{\uparrow})$ whose distribution is uniquely determined by:

1. For each $z \in \mathbb{R}^2$, almost surely there exists a unique $\pi_z \in \Pi^{\uparrow}$ such that $\mathcal{W}(z) = \{\pi_z\}$.
2. For each $z_1, \ldots, z_n \in \mathbb{R}^2$, the paths $(\pi_{z_1}, \ldots, \pi_{z_n})$ are distributed as coalescing Brownian motions starting from $z_1, \ldots, z_n$.
3. For each countable dense set $\mathcal{D} \subset \mathbb{R}^2$, almost surely $\mathcal{W} = \overline{\mathcal{W}(\mathcal{D})}$.

Here $\mathcal{W}(\mathcal{D}) := \{\pi \in \mathcal{W} : \pi \text{ starts in } \mathcal{D}\}$.

The Brownian web



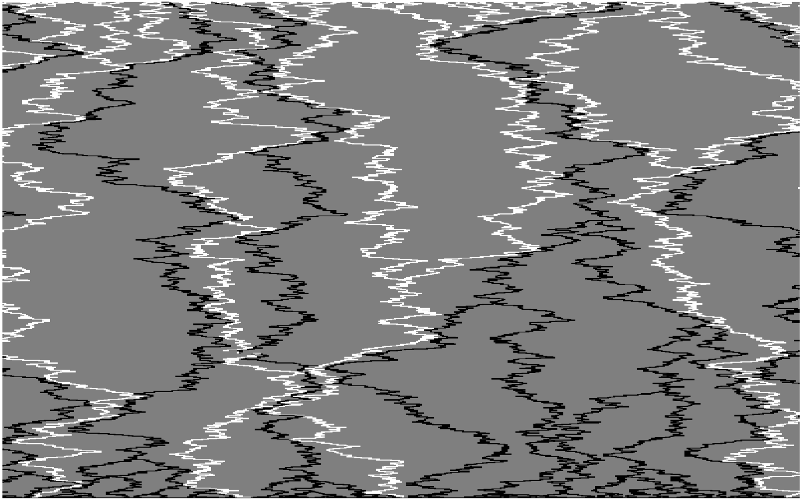
Each arrow configuration defines a *dual* arrow configuration.

The Brownian web

Associated to each Brownian web $\mathcal{W}$, there is a *dual* Brownian web $\hat{\mathcal{W}}$ that is a.s. uniquely determined by $\mathcal{W}$ and equally distributed with $\mathcal{W}$ after a rotation over 180°.

Dual paths reflect off forward paths with Skorohod reflection.

The Brownian web



Forward paths (black) and dual paths (white) starting at two fixed times.

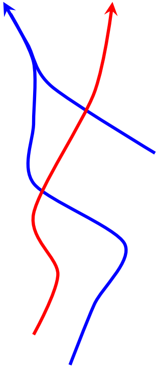
The Brownian net

In case of a left-right Brownian web $(\mathcal{W}^{\rm l},\mathcal{W}^{\rm r})$, the dual webs $(\hat{\mathcal{W}}^{\rm l},\hat{\mathcal{W}}^{\rm r})$ are equally distributed with the forward webs after a rotation over 180°.

The paper *The Brownian net* gives three ways to construct a Brownian net $\mathcal{N}$ from a left-right Brownian web and its dual $(\mathcal{W}^{\rm l},\mathcal{W}^{\rm r},\hat{\mathcal{W}}^{\rm l},\hat{\mathcal{W}}^{\rm r})$:

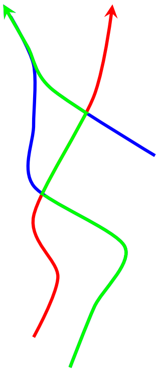
1. The hopping construction.
2. The wedge construction.
3. The mesh construction.

The hopping construction



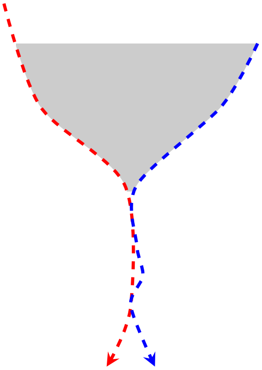
$$\mathcal{N} = \overline{\mathcal{H}_{\rm hop}(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})}.$$

The hopping construction



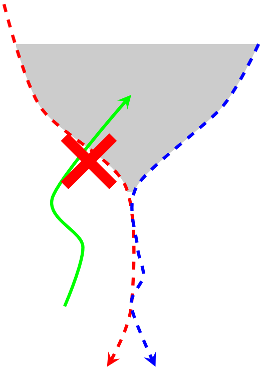
$$\mathcal{N} = \overline{\mathcal{H}_{\rm hop}(\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})}.$$

The wedge construction



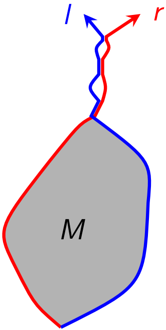
A *wedge* is the open area between a dual
left and right path above their first meeting point.

The wedge construction



$\mathcal{N} = \{\pi \in \Pi^{\uparrow} : \pi$ does not enter wedges of $(\mathcal{W}^{\mathrm{l}}, \mathcal{W}^{\mathrm{r}})\}$.

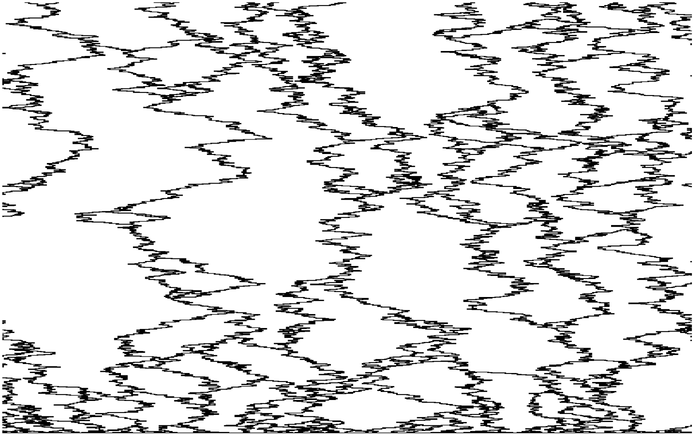
The mesh construction



A *mesh* M is the open area enclosed by
a left and right path starting from the same point,
that are initially ordered the "wrong" way.

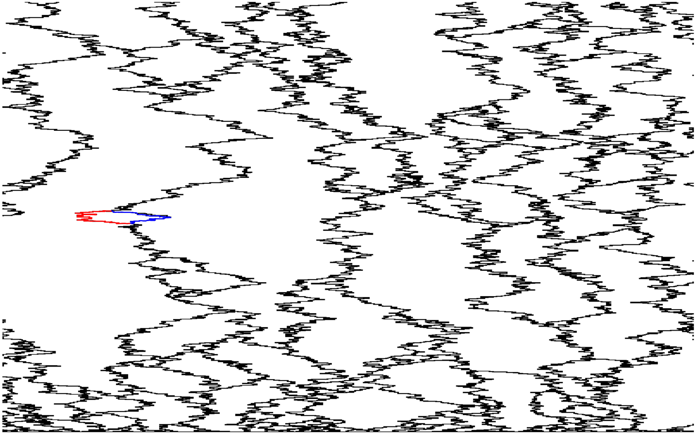
$$\mathcal{N} = \{\pi \in \Pi^{\uparrow} : \pi \text{ does not enter meshes of } (\mathcal{W}^{\rm l}, \mathcal{W}^{\rm r})\}.$$

Meshes



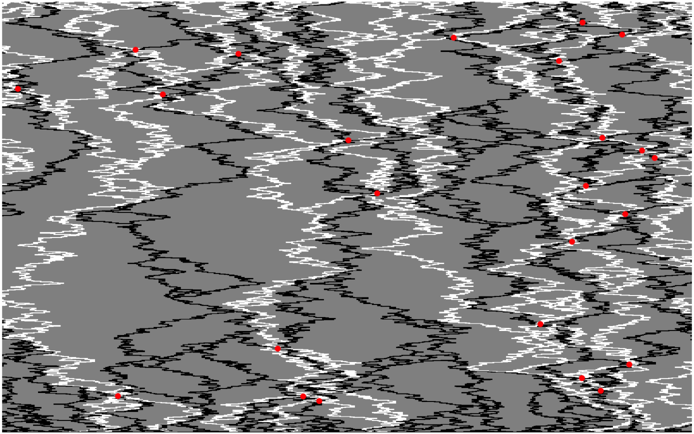
Paths in the Brownian net starting at time zero.

Meshes



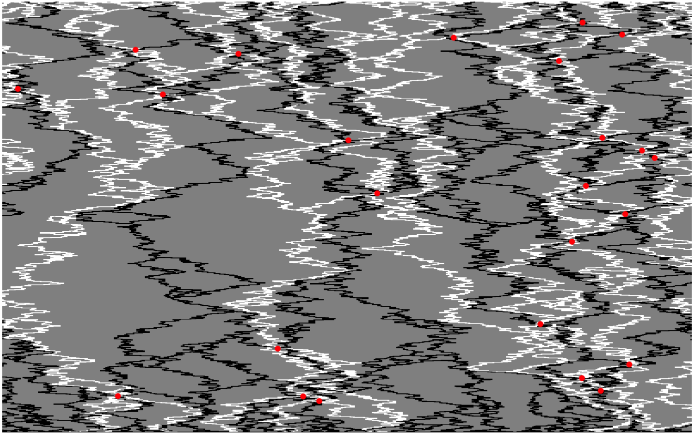
The connected components
of the complement of all paths are meshes.

Meshes



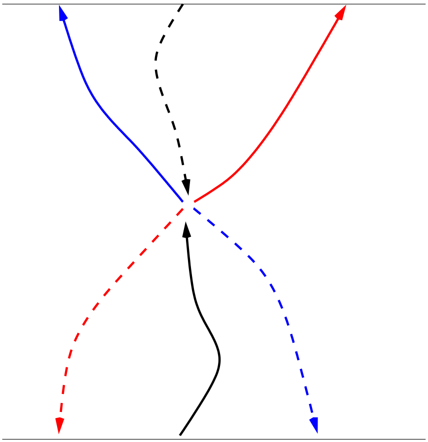
Paths in the Brownian net and the dual Brownian net started from fixed times.

Meshes



Dual net paths exit forward meshes via their bottom points.
These are the relevant separation points

Relevant separation points



A relevant separation point.

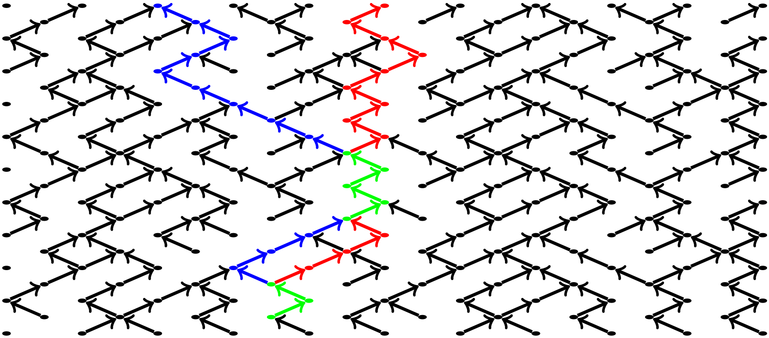
Convergence

The original proof that arrow configurations with a small branching rate converge to the Brownian net is based on the *hopping* and *wedge* constructions:

- Tightness of rescaled discrete nets follows from tightness of the left and right webs.
- Each cluster point $\mathcal{N}$ satisfies $\mathcal{N}_{\rm hop} \subset \mathcal{N} \subset \mathcal{N}_{\rm wedge}$.

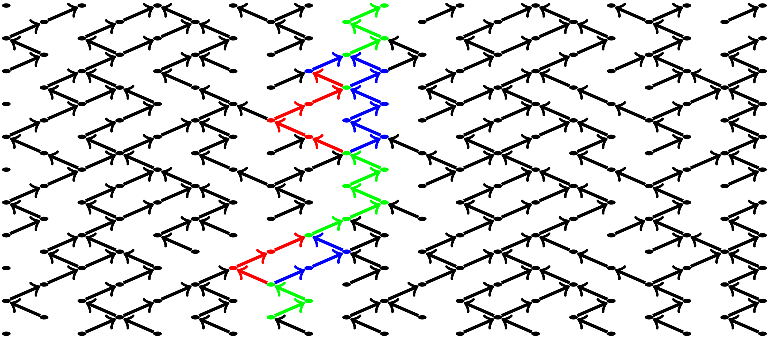
Open Problem: Prove convergence based on *meshes* rather than *wedges*.

Mesh-paths



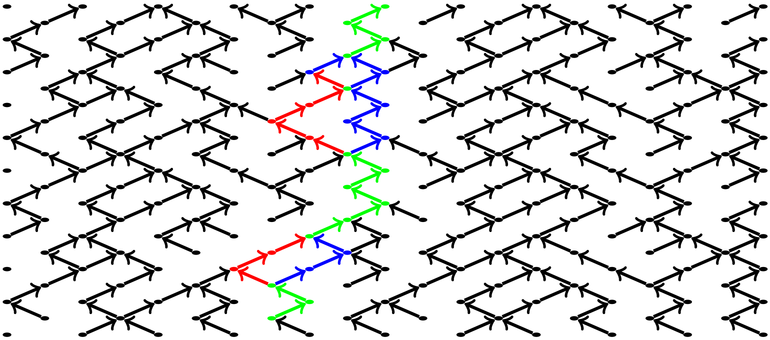
Once a left and right path separate,
they try to get away from each other.

Mesh-paths



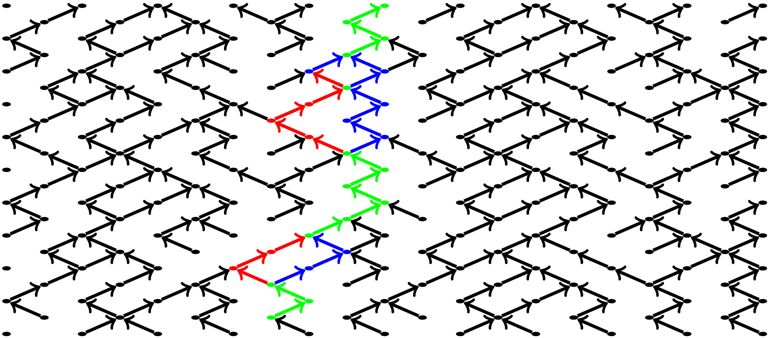
Instead, one can look at pairs of paths that once they separate, try to get back together as soon as possible.

Mesh-paths



The open areas enclosed by such a *mesh-pair* are meshes.

Mesh-paths



Open problem: Characterise the joint law of the scaling limit of several mesh-pairs.

The branching-coalescing point set

Let $\mathrm{Clos}(\mathbb{R})$ be the space of closed subsets of $\mathbb{R}$.
For any $A \in \mathrm{Clos}(\mathbb{R})$, setting

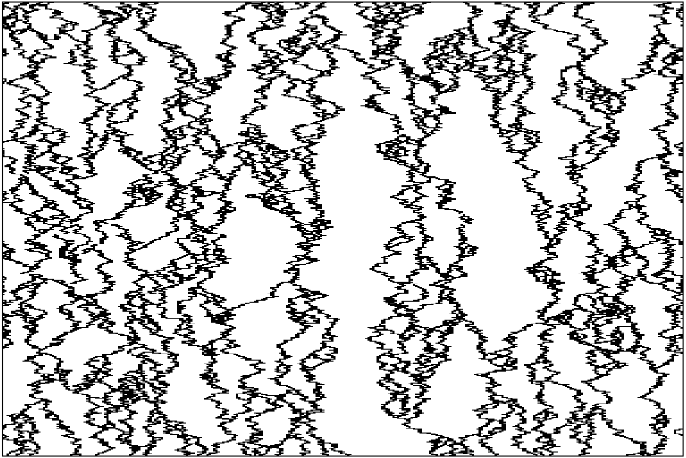
$$\xi_t^A := \big\{\pi(t) : \pi \in \mathcal{N}\big(A \times \{0\}\big)\big\}$$

defines a Markov process $(\xi_t^A)_{t\geq 0}$ with values in $\mathrm{Clos}(\mathbb{R})$, the *branching-coalescing point set*.

- ξ_t^A is a.s. locally finite for deterministic $t > 0$.
- Poisson point set with intensity 2 is reversible.
- There are random times when ξ_t^A has no isolated points.
- Feller process with compact state space $\mathrm{Clos}(\mathbb{R})$.

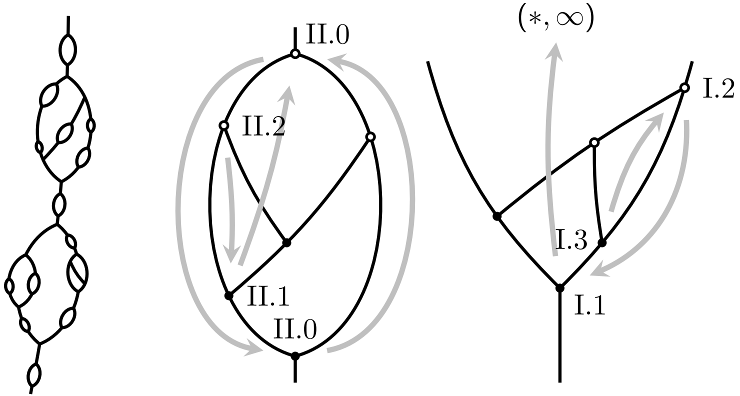
Open problem: Generator characterisation of $(\xi_t^A)_{t\geq 0}$.

The branching-coalescing point set



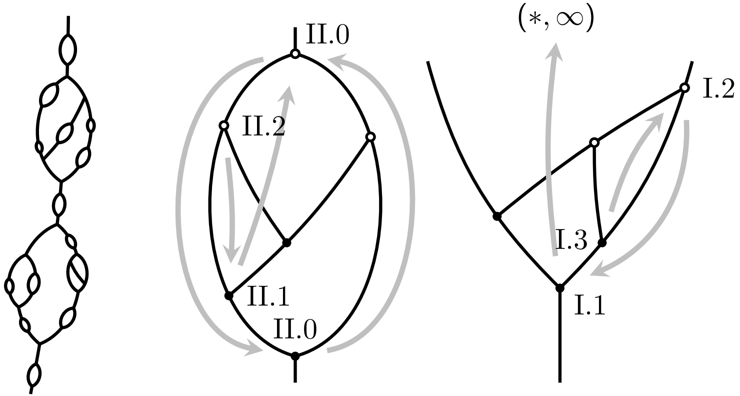
The backbone of the Brownian net.

The branching-coalescing point set



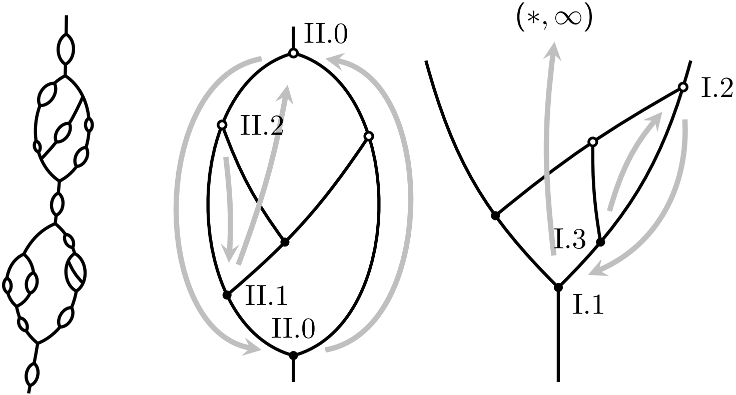
For a separation point z, let $\phi(z)$ be the first meeting point of the left-most and right-most paths starting at z.

The branching-coalescing point set



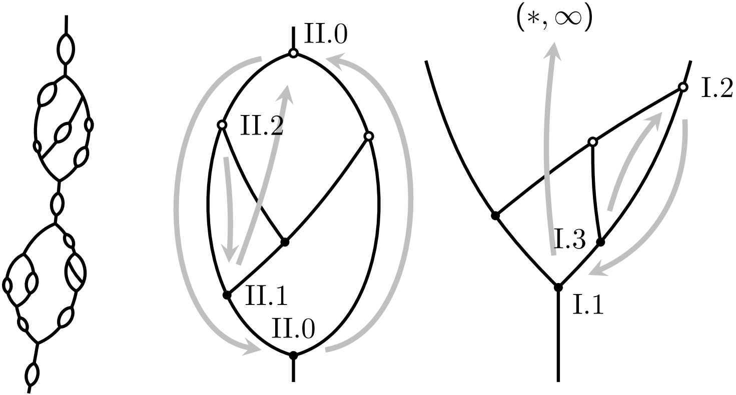
For a meeting point z, define $\phi(z)$ analogously by turning the picture upside down.

The branching-coalescing point set



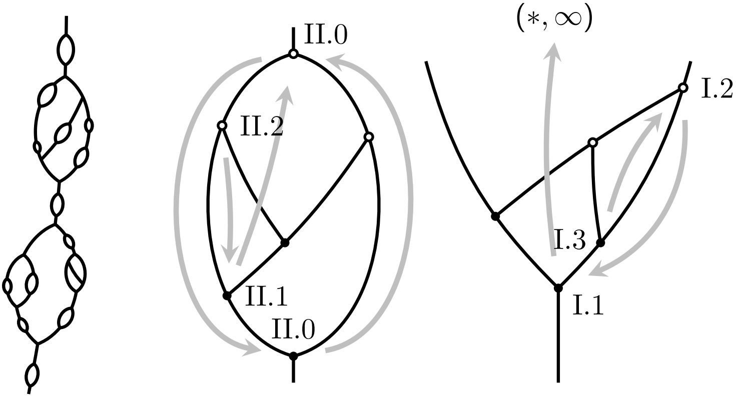
Call a point of type $\mathrm{I}.n$ if $z \mapsto \phi(z) \mapsto \phi^2(x) \mapsto \cdots$
reaches $(*, \pm\infty)$ after n steps.

The branching-coalescing point set



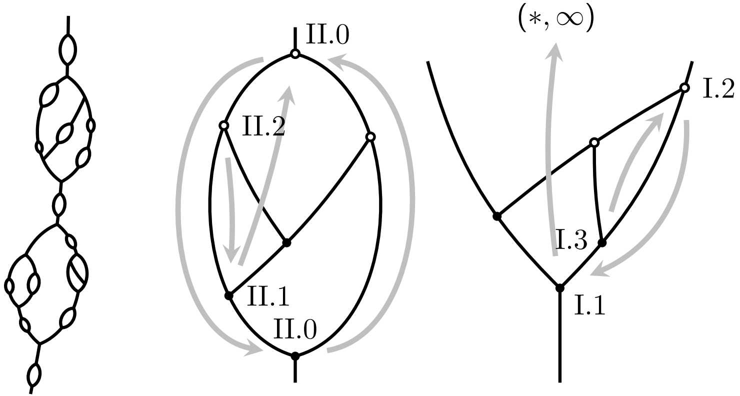
Call a point of type II.n if $z \mapsto \phi(z) \mapsto \phi^2(x) \mapsto \cdots$ reaches a point z' with $\phi^2(z') = z'$ after n steps.

The branching-coalescing point set



Bubble hypothesis: All separation and meeting points are of type I.n or II.n for some finite n.

The branching-coalescing point set



Bubble complexity hypothesis: Points of types II.0 and II.1 are dense on the backbone and all others are locally finite.

The branching-coalescing point set

Open problem: Relation to Feynman diagrams? Only two diagrams need to be renormalised?

Open problem: Relation to a quantum field theory?