Augmented Brownian webs as the scaling limit of non-simple coalescing random walks

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joint with Nic Freeman

Discrete webs

Let p be a probability law on $\mathbb{Z}$ and let $\mathrm{supp}(p) := \{k \in \mathbb{Z} : p(k) > 0\}$.

Assume that:

(i) (*Symmetry*) $p(k) = p(-k)$.

(ii) (*Irreducibility*) $\forall k \in \mathbb{Z},\ \exists k_1, \ldots, k_n \in \mathrm{supp}(p)$ s.t. $k_1 + \cdots + k_n = k$.

(iii) (*Aperiodicity*) The least common divisor of the set $\big\{n \geq 1 : \exists k_1, \ldots, k_n \in \mathrm{supp}(p) \text{ s.t. } k_1 + \cdots + k_n = 0\big\}$ is one.

(iv) (*Finite variance*) $\sigma^2 := \sum_{k \in \mathbb{Z}} p(k)k^2 < \infty$.

Let $\omega = \big(\omega(x, s)\big)_{(x,s) \in \mathbb{Z}^2}$ be i.i.d. with law p.

Discrete webs

For each $z = (x, s) \in \mathbb{Z}^2$, there exists a unique path π_z with domain $I(\pi_z) := \{s, s+1, \ldots\}$ such that

$$\pi_z(s) = x \quad \text{and} \quad \pi(t+1) = \pi(t) + \omega\big(\pi_z(t), t\big) \qquad (t \geq s).$$

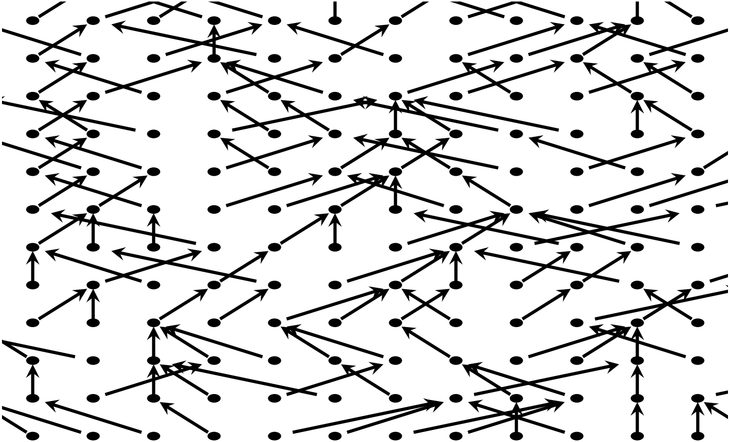
For each $s \in \overline{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, \infty\}$ we define trivial paths $\pi^{\pm}_s$ with domain $I(\pi^{\pm}_s) := \{t \in \mathbb{Z} : t > s\}$ by

$$\pi^{\pm}_s(t) := \pm\infty \qquad (t \in I(\pi^{\pm}_s)).$$

Lemma Almost surely, $\mathcal{U} := \big\{\pi_z : z \in \mathbb{Z}^2\big\} \cup \{\pi^{\pm}_s : s \in \overline{\mathbb{Z}}\}$ is a compact subset of the space Π_c of "continuous" paths, equipped with the Brownian web topology.

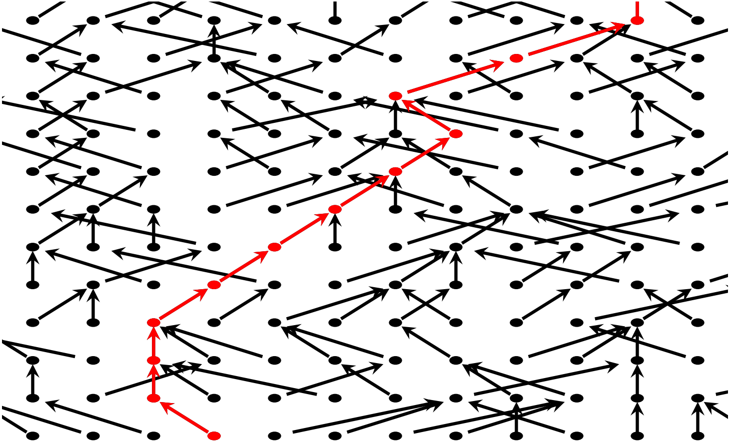
With the formalism of my previous talk, no need to interpolate!

Discrete webs



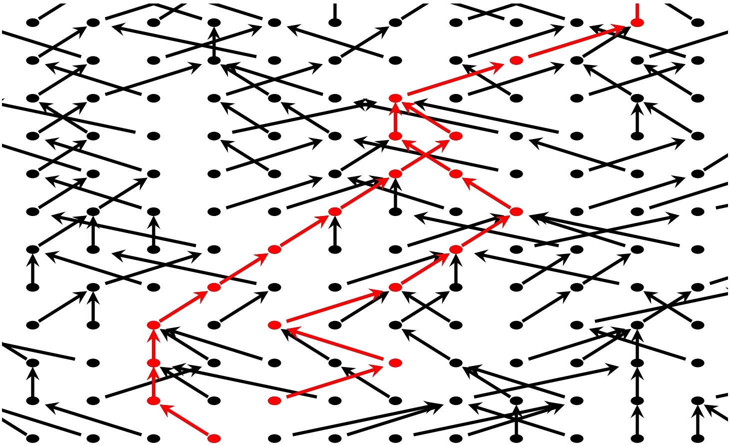
An arrow configuration ω.

Discrete webs



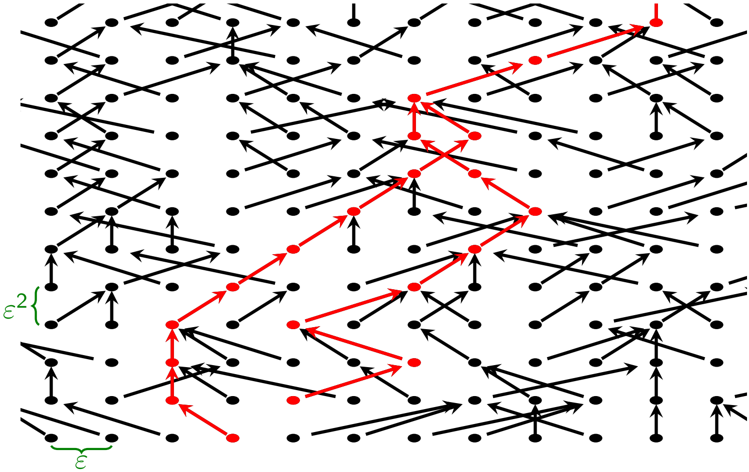
A path π_z (interpolated for better visibility).

Discrete webs



Paths coalesce as soon as they meet.

Discrete webs



We rescale space by ε and time by ε^2.

Discrete webs

We define diffusive scaling maps by

$$S_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t).$$

We extend S_ε continuously to the squeezed space $\mathcal{R}(\overline{\mathbb{R}})$.

We identify a path with its closed graph and set

$$S_\varepsilon(\pi) := \big\{S_\varepsilon(z) : z \in \pi\big\} \quad \text{and} \quad S_\varepsilon(\mathcal{U}) := \big\{S_\varepsilon(\pi) : \pi \in \mathcal{U}\big\}.$$

We are interested in the limit of $\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big]$ as $\varepsilon \to 0$.

The Brownian web

The Brownian web with diffusion rate σ is the unique (in law) random variable $\mathcal{W}^\sigma$ with values in $\mathcal{K}_+(\Pi^\uparrow_{\rm c})$ such that:

(i) In each deterministic point $z \in \mathbb{R}^2$ there almost surely starts precisely one path $\pi_z \in \mathcal{W}^\sigma$.

(ii) The paths $\pi_{z_1}, \ldots, \pi_{z_k}$ starting in a finite collection $z_1, \ldots, z_k \in \mathbb{R}^2$ of deterministic points are distributed as coalescing Brownian motions with diffusion rate σ^2.

(iii) For each deterministic countable dense set $\mathcal{D} \subset \mathbb{R}^d$, the Brownian web $\mathcal{W}^\sigma$ is the closure of the set $\mathcal{W}^\sigma(\mathcal{D}) := \{\pi_z : z \in \mathcal{D}\}$ of paths starting in $\mathcal{D}$.

Standard Brownian web $\mathcal{W} := \mathcal{W}^1$.

Convergence to the Brownian web

Theorem [Freeman, S. '25] Assume

$$n^3 P(n) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{with} \quad P(n) := \sum_{k=n}^{\infty} p(k) \quad (n \geq 0).$$

Then

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^\sigma \in \cdot\,\big].$$

Proved earlier **[Belhaouari, Mountford, Sun, and Valle '06]** under the assumption $\alpha > 3$, where

$$\alpha := \sup\Big\{\beta \geq 0 : \sum_{k\in\mathbb{Z}} p(k)|k|^\beta < \infty\Big\}$$

$$= \sup\Big\{\beta \geq 0 : n^\beta P(n) \underset{n\to\infty}{\longrightarrow} 0\Big\}.$$

Convergence to the Brownian web

The condition $n^3 P(n) \underset{n\to\infty}{\longrightarrow} 0$ is sharp. Set

$$X := \big\{(x,y,t) : (x,t) \in \mathbb{Z}^2,\ y = x + \omega(x,t)\big\},$$
$$S_\varepsilon(X) := \big\{(\varepsilon x, \varepsilon y, \varepsilon^2 t) : (x,y,t) \in X\big\}.$$

Assume $n^3 P(n) \underset{n\to\infty}{\longrightarrow} c > 0$. Then

$$\mathbb{P}\big[S_\varepsilon(X) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\Xi_c \in \cdot\,\big],$$

where Ξ_c is a Poisson point process on $\big\{(x,y,t) \in \mathbb{R}^3 : x \neq y\big\}$ with intensity measure $c\mu$, where

$$\mu\big(\mathrm{d}(x,y,t)\big) := |x-y|^{-4}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

The augmented Brownian web

For each $x \in \mathbb{R}$ and $\pi \in \Pi^{\uparrow}$ with starting time $\sigma_\pi = s \in \mathbb{R}$, we define $\pi^{(x)} \in \Pi^{\uparrow}$ with domain $I(\pi^{(x)}) := I(\pi)$ by

$$\pi^{(x)}(s-) := x \quad \text{and} \quad \pi^{(x)}(t\pm) := \pi(t\pm) \quad \text{in all other cases.}$$

Let $\mathcal{W}^\sigma$ be a Brownian web and let Ξ_c be an independent Poisson point process on $\big\{(x, y, t) \in \mathbb{R}^3 : x \neq y\big\}$ with intensity $c\mu$.

Then we define the *augmented Brownian web* $\mathcal{W}^{\sigma,c}$ by

$$\mathcal{W}^{\sigma,c} := \mathcal{W}^\sigma \cup \big\{\pi^{(x)}_{(y,t)} : (x, y, t) \in \Xi_c\big\},$$

where $\pi_{(y,t)}$ is the a.s. unique path in $\mathcal{W}^\sigma$ starting at (y, t). Similarly, we define

$$\mathcal{W}^{\sigma,\infty} := \mathcal{W}^\sigma \cup \big\{\pi^{(x)} : \pi \in \mathcal{W}^\sigma,\ \sigma_\pi \in \mathbb{R}\big\}.$$

Convergence to the Brownian web

Recall

$$\alpha := \sup\Big\{\beta \geq 0 : \sum_{k\in\mathbb{Z}} p(k)|k|^{\beta} < \infty\Big\}$$

Theorem [Freeman, S. '25] Assume $\alpha > 9/4$ and

$$n^3 P(n) \underset{n\to\infty}{\longrightarrow} c \in [0,\infty] \quad \text{with} \quad P(n) := \sum_{k=n}^{\infty} p(k) \quad (n \geq 0).$$

Then

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^{\sigma,c} \in \cdot\,\big],$$

where $\Rightarrow$ denotes weak convergence on Π with respect to Skorohod's J1 topology, and paths in $\mathcal{U}$ are *not* interpolated.

Convergence to the Brownian web

The condition $\alpha > 9/4$ is sharp in the sense that for $\alpha < 9/4$ the statement of the theorem is false.

The reason is that for $\alpha < 9/4$, in the diffusive scaling limit, there appear paths that make *two* macroscopic jumps near their starting time, which contradicts convergence in Skorohod's J1 topology.

In the remainder of this talk,
I will explain where the $9/4$ comes from.

Presumably, it would be possible to prove convergence statements also for $\alpha < 9/4$ but this would require significant work to define an appropriate topology and also the limit object becomes significantly more involved.

Removing the symmetry assumption on the jump distribution p is much easier but a notational nuisance.

A multiscale decomposition of discrete webs

Define sublattices $\mathbb{Z}^2 = \Lambda^0 \supset \Lambda^1 \supset \Lambda^2 \supset \cdots$ by

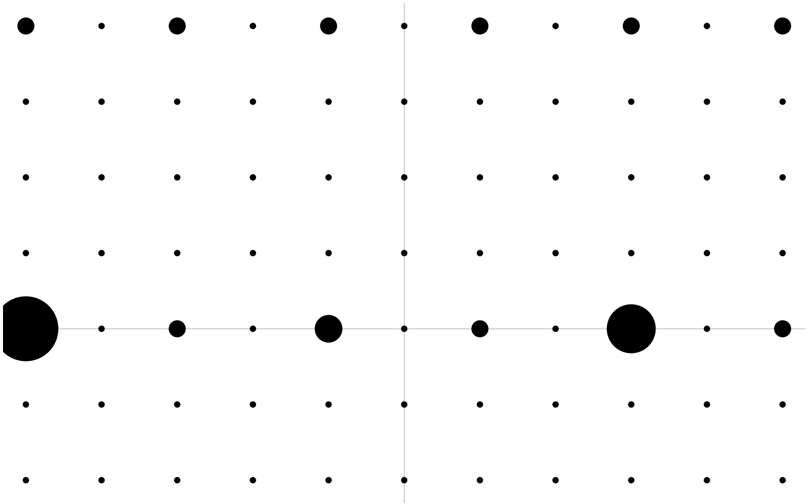
$$\Lambda^d := \big\{(r_d + 2^d x, 4^d t) : (x,t) \in \mathbb{Z}^d\big\} \quad \text{with} \quad r_d := \sum_{i=0}^{d-1}(-2)^i.$$

For $(x,s),(y,t) \in \mathbb{Z}^2$, set

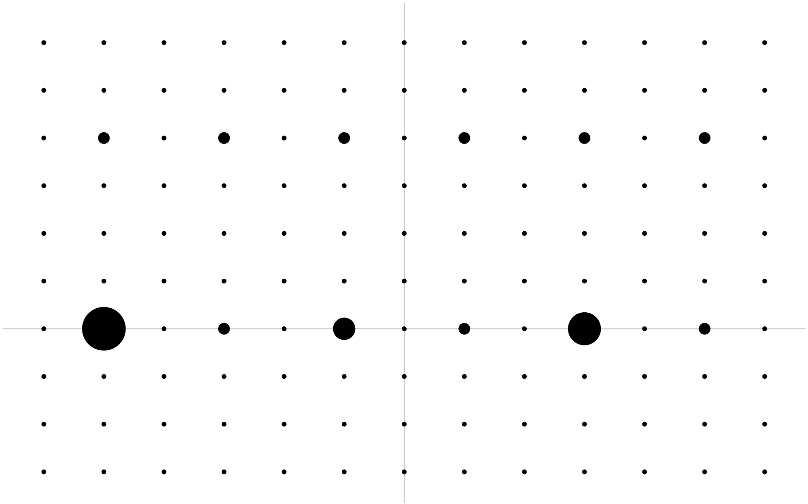
$$\begin{aligned} \ell_0(x,s) &:= \sup\big\{d \geq 0 : (x,s) \in \Lambda^d\big\}, \\ \ell(y,t) &:= \sup\big\{d \geq 0 : \exists (x,s) \in \Lambda^d \text{ s.t. } s \leq t,\ \pi_{(x,s)}(t) = y\big\}. \end{aligned}$$

Then $I(\pi) \ni t \mapsto \ell\big(\pi(t), t\big)$ is nondecreasing for each $\pi \in \mathcal{U}$.

A multiscale decomposition of discrete webs



A multiscale decomposition of discrete webs



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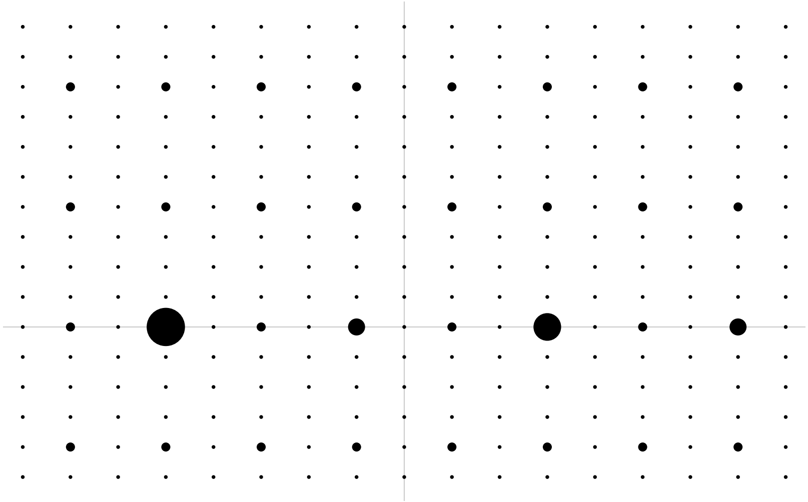
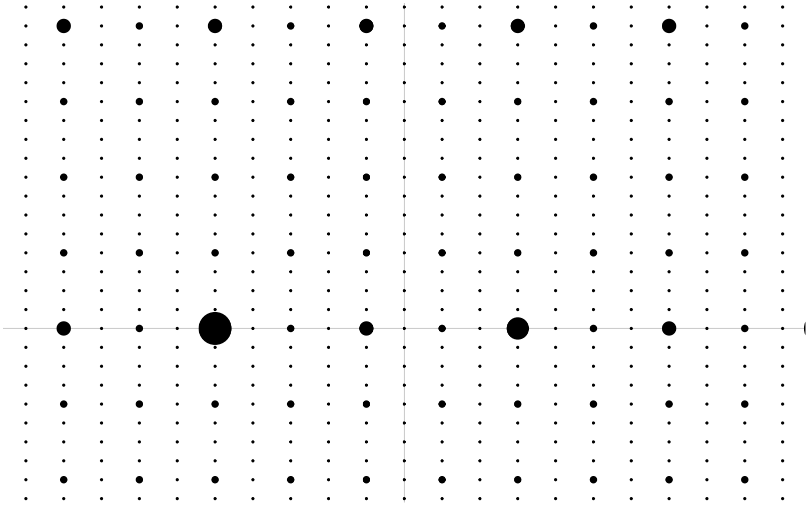


A multiscale decomposition of discrete webs



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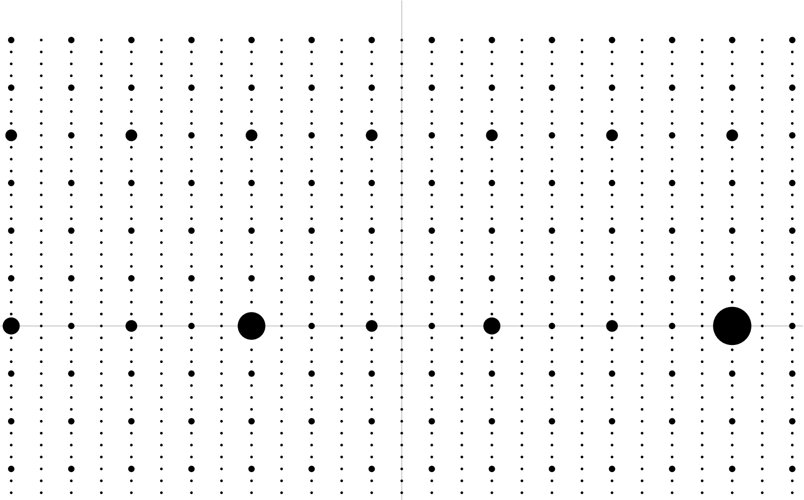
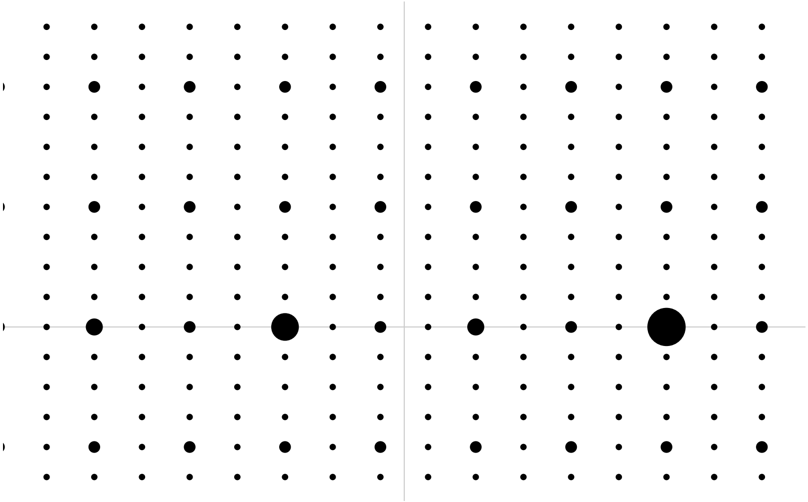
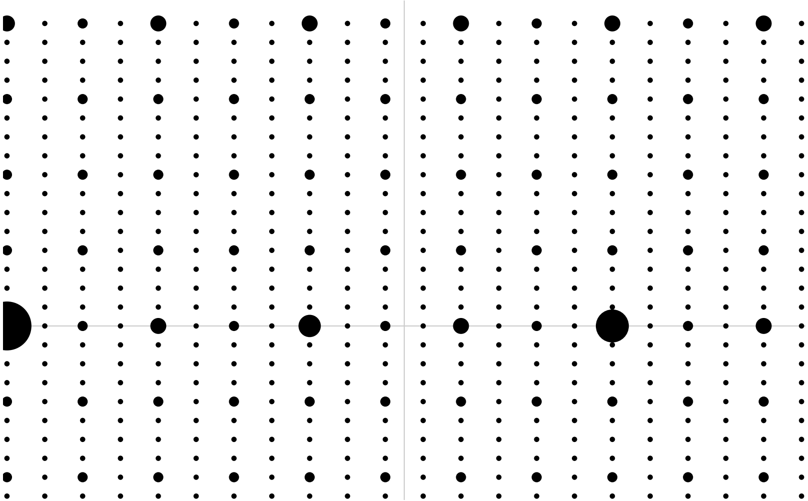


A multiscale decomposition of discrete webs



A multiscale decomposition of discrete webs



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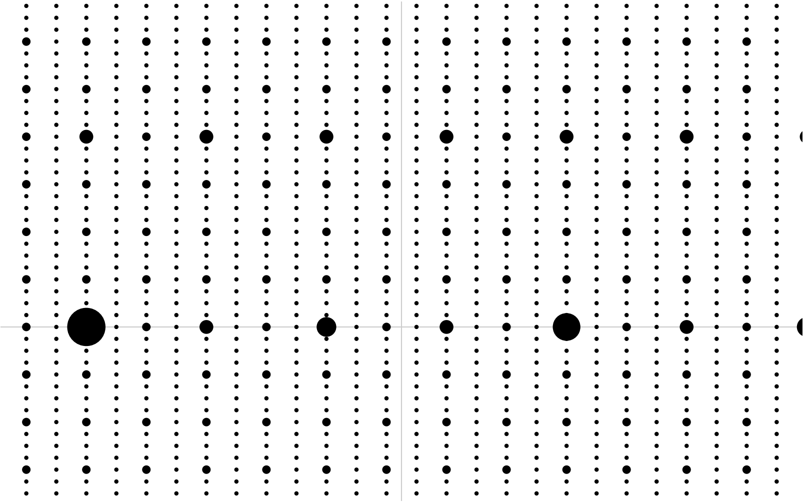

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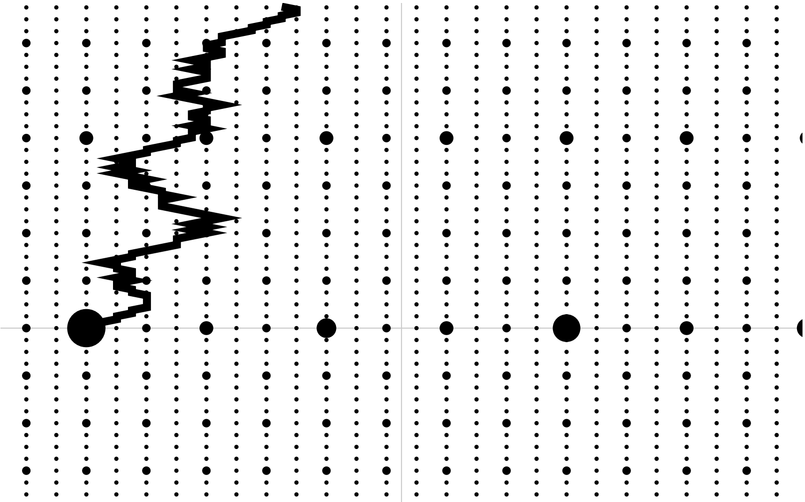

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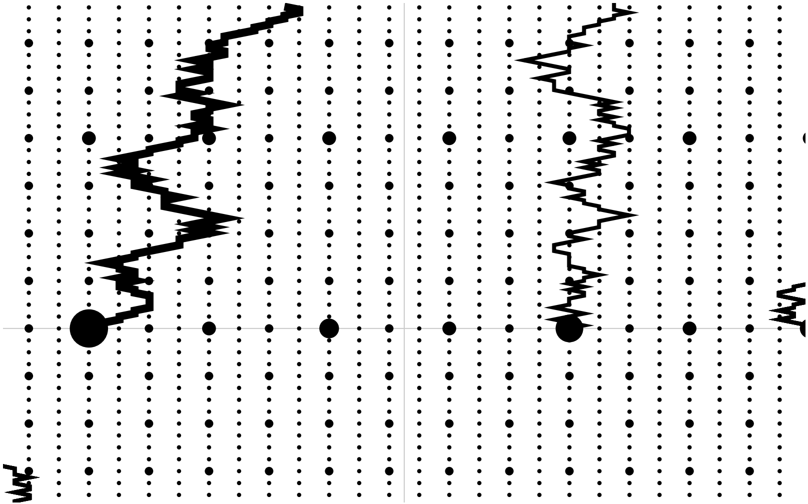
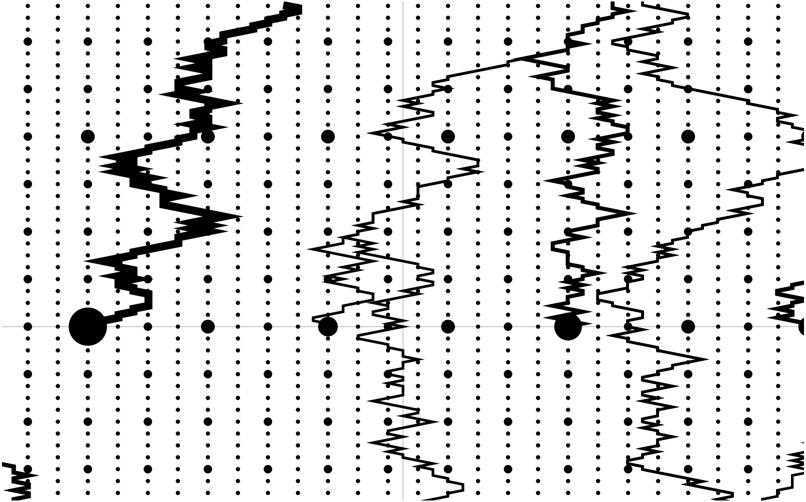


A multiscale decomposition of discrete webs



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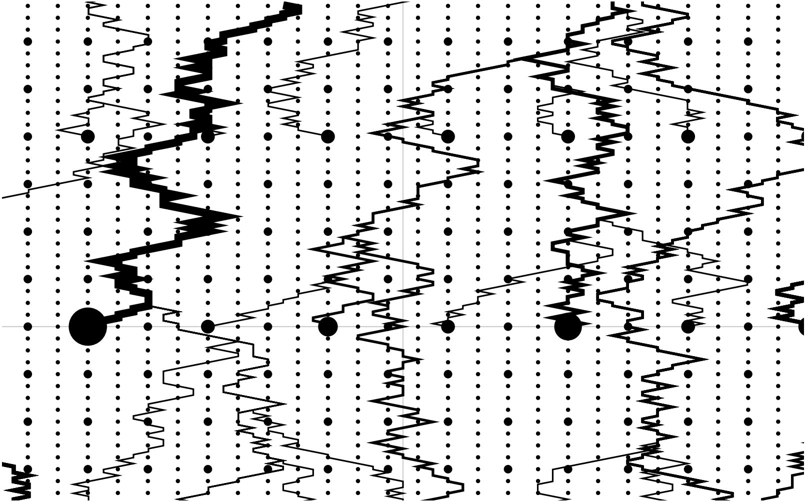

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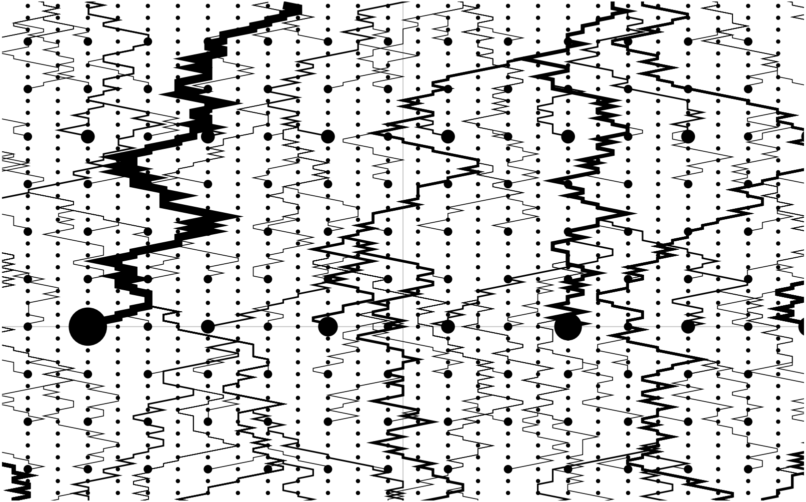


A multiscale decomposition of discrete webs

Recall that $\ell(y,t)$ is the level of the highest-level path that passes through (y,t). We let

$$\tau_d(z) := \inf\big\{t \geq 0 : \ell\big(\pi_z(\sigma_{\pi_z}+t),t\big) \geq d\big\}$$

denote the time before the path started at z coalesces with a path of level d or higher.

Lemma There exists constants $C<\infty$ and $\lambda>0$ such that

$$\mathbb{P}\big[\tau_d(z) > t4^d\big] \leq Ce^{-\lambda t} \qquad (t \geq 0,\ d \geq 1,\ z \in \mathbb{Z}^2).$$

A multiscale decomposition of discrete webs

Recall $S_\varepsilon(x,s) := (\varepsilon x, \varepsilon^2 s)$. Let

$$\Lambda_D := \big\{(x,s) \in \mathbb{Z}^2 : 0 \leq x < 2^D,\ 0 \leq t < 4^D\big\}$$

denote the *diffusive window* and let $\Lambda^d_D := \Lambda^d \cap \Lambda_D$. The number of points of level d or higher in the diffusive window is

$$|\Lambda^d_D| = 8^{D-d} \qquad (0 \leq d \leq D).$$

Because of the exponential tails, for each $\varepsilon > 0$,
there exists a constant $C < \infty$ such that with probability $\geq 1 - \varepsilon$, *all* paths started in Λ^d_D *level up* within a time $\leq C(D-d)4^d$, uniformly in D and d.

Side remark Let

$$\rho_t := \mathbb{P}\big[\exists \pi \in \mathcal{U} \text{ s.t. } \sigma_\pi = 0,\ \pi(t) = 0\big].$$

denote the density of coalescing random walks, starting from each point in space. Using the previous lemma, it should be easy to show that

$$\rho_t \asymp t^{-1/2} \quad \text{as } t \to \infty.$$

In fact, the method should also work for random walks in the domain of attraction of an α-stable Lévy process, with $1 < \alpha < 2$. In this case,

$$\rho_t \asymp t^{-1/\alpha} \quad \text{as } t \to \infty,$$

the sublattices should be defined differently, corresponding to the scaling map $S^\alpha_\varepsilon(x, s) := (\varepsilon x, \varepsilon^\alpha s)$, and one should use that α-stable Lévy processes with $1 < \alpha < 2$ are point recurrent.

Thinned-out lattices

Let $\pi|_r := \{(x,t) \in \pi : t \geq \sigma_\pi + r\}$ and $\mathcal{U}|_r := \{\pi|_r : \pi \in \mathcal{U}\}$.

Recall $\alpha := \sup\{\beta \geq 0 : \sum_{k\in\mathbb{Z}} p(k)|k|^\beta < \infty\}$.

Theorem Assume $\alpha > 2$ and choose $3 - \alpha < \beta < 1$. Then

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}|_{\varepsilon^{-2\beta}}) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^\sigma \in \cdot\,\big].$$

Proof sketch Recall

$$\begin{aligned}\ell_0(x,s) &:= \sup\{d \geq 0 : (x,s) \in \Lambda^d\},\\ \ell(y,t) &:= \sup\{d \geq 0 : \exists (x,s) \in \Lambda^d \text{ s.t. } s \leq t,\ \pi_{(x,s)}(t) = y\}.\end{aligned}$$

Because of the lemma,

$$\big|\{z \in \Lambda_D : \ell(z) = d\}\big| \approx 8^{D-d} \cdot 4^d = 2^{3D-d}.$$

Thinned-out lattices

It follows that $\big|\{z \in \Lambda_D : \ell(z) = d,\ |\omega(z)| \geq 2^{\theta D}\}\big|$

$$\approx 2^{3D-d} \cdot 2^{-\alpha\theta D} = 2^{(3-\alpha\theta)D-d}.$$

This quantity is of order one for $\theta = (3D-d)/(\alpha D)$, so

$$\sup\big\{|\omega(z)| : z \in \Lambda_D,\ \ell(z) \geq \beta D\big\} \approx 2^{(3-\beta)D/\alpha}.$$

We have $2^{(3-\beta)D/\alpha} \ll 2^D$ as long as
$(3-\beta)/\alpha < 1 \quad \Leftrightarrow \quad 3-\alpha < \beta$, so

$$\mathbb{P}\big[S_{2^{-D}}(\mathcal{U}(\Lambda^{\beta D}) \in \cdot\,\big] \underset{D\to\infty}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^\sigma \in \cdot\,\big].$$

Setting $\varepsilon = 2^{-D}$, using that all paths reach level βD within a time of order $4^{\beta D} = \varepsilon^{-2\beta}$, the claim follows. ∎

Two macroscopic jumps

Let $N(D,d,d')$ denote the number of paths starting in Λ_D^d that make *two* jumps of size $\geq 2^D$, the first one while they are at level d and the second one when they have reached level $d' \geq d$. Then

$$\mathbb{E}\big[N(D,d,d')\big] \approx 8^{D-d} \cdot 4^d 2^{-\alpha D} \cdot 4^{d'} 2^{-\alpha D} = 2^{(3-2\alpha)D-d+2d'}.$$

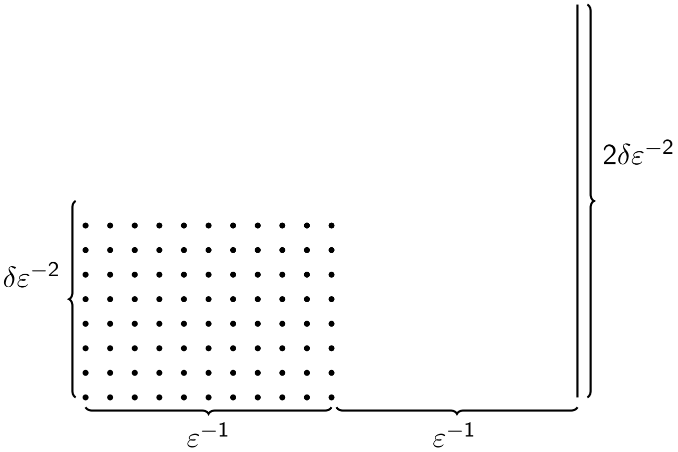
By our earlier argument we can restrict to $d' \leq (3-\alpha)D$.

The number of paths starting in Λ_D and making two jumps $\geq 2^D$ is

$$\approx \sum_{d=0}^{(3-\alpha)D} \sum_{d'=d}^{(3-\alpha)D} 2^{(3-2\alpha)D-d+2d'} \approx 2^{(9-4\alpha)D},$$

which comes from the term with $d = 0$ and $d' = (3-\alpha)D$.
As long as $\alpha > 9/4$, this tends to zero as $D \to \infty$. ∎

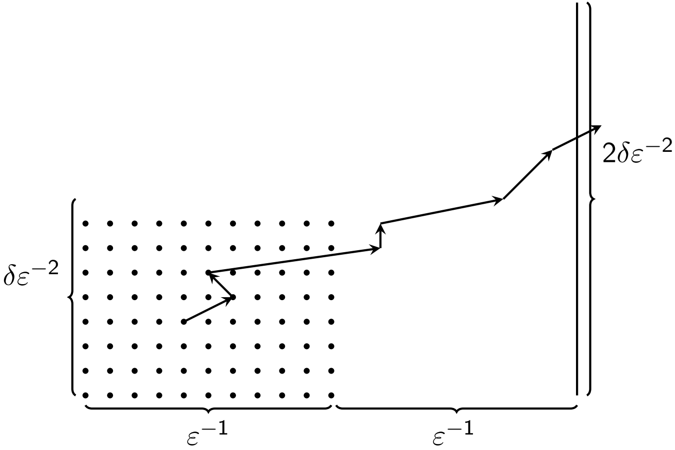
A multiscale argument



[Belhaouari, Mountford, Sun, & Valle '06]

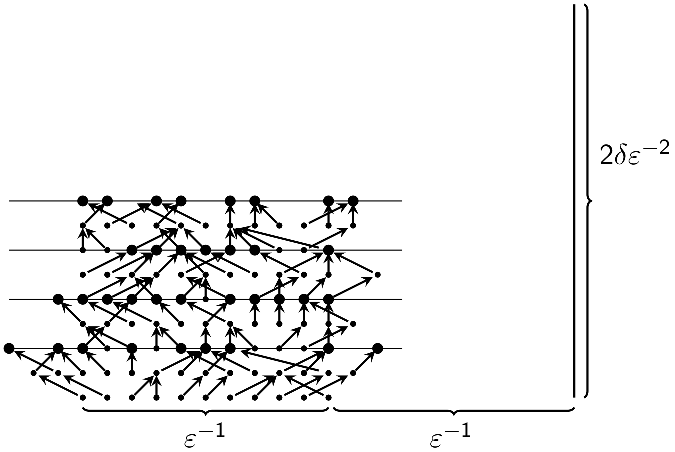
used a different multiscale argument for discrete webs.

A multiscale argument



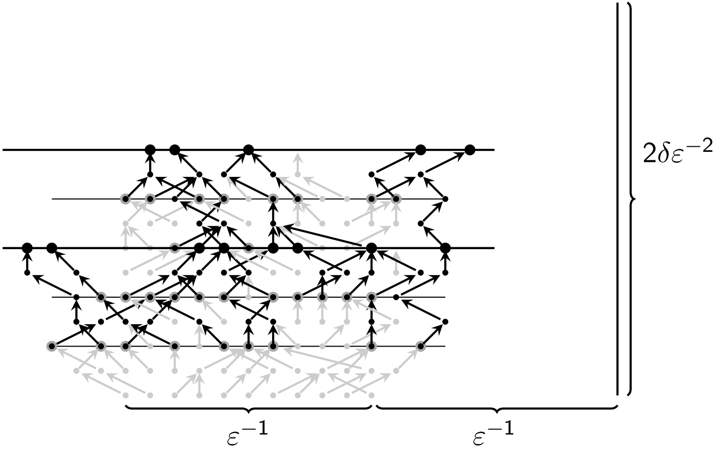
For tightness one needs to show that the probability of this event tends to zero for $\delta \to 0$, uniformly in ε.

A multiscale argument



In each step, they let the coalescing random walks of two subsequent times evolve until the next time.

A multiscale argument



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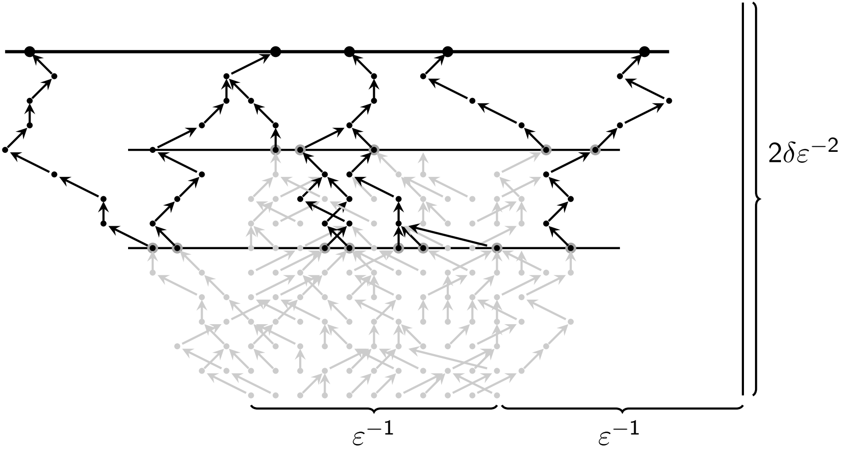

In each step, they let the coalescing random walks of two subsequent times evolve until the next time.

A multiscale argument

By controlling the number of particles and their displacement in each step, Belhaouari, Mountford, Sun, & Valle in 2006 were able to control the maximal displacement of all paths started in the block of size $\varepsilon^{-1} \times \delta\varepsilon^{-2}$.

The argument is a bit lossy, which is why they needed the condition $\alpha > 3$, which is slightly weaker than our optimal condition

$$n^3 P(n) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{with} \quad P(n) := \sum_{k=n}^{\infty} p(k) \quad (n \geq 0).$$

The advantage of our multiscale argument is that one first controls the *time* till coalescence, which is easy, and only later has to care about the *displacement* before coalescence.