Topologies on sets of paths

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joint with Nic Freeman

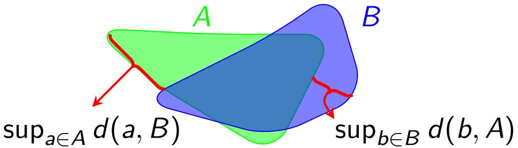
The Hausdorff metric

Let $(\mathcal{X}, d)$ be a metric space.
Let $\mathcal{K}_+(\mathcal{X})$ be the set of nonempty compact subsets of $\mathcal{X}$.
The *Hausdorff metric* on $\mathcal{K}_+(\mathcal{X})$ is defined as

$$d_{\mathrm{H}}(A, B) := \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A)$$

where

$$d(a, B) := \inf_{b \in B} d(a, b).$$



The Hausdorff metric

A *correspondence* between two sets A_1, A_2 is a set $R \subset A_1 \times A_2$ such that

$$\forall x_1 \in A_1\ \exists x_2 \in A_2 \text{ s.t. } (x_1, x_2) \in R,$$
$$\forall x_2 \in A_2\ \exists x_1 \in A_1 \text{ s.t. } (x_1, x_2) \in R.$$

Let $\mathrm{Cor}(A_1, A_2)$ denote the set of all correspondences between A_1 and A_2.

$$d_{\mathrm{H}}(K_1, K_2) = \inf_{R \in \mathrm{Cor}(K_1, K_2)} \sup_{(x_1, x_2) \in R} d(x_1, x_2).$$

The Hausdorff topology

For $K_n, K \in \mathcal{K}_+(\mathcal{X})$ one has $d_{\rm H}(K_n, K) \to 0$ iff

(i) $\exists C \in \mathcal{K}_+(\mathcal{X})$ s.t. $K_n \subset C \quad \forall n$,

(ii) $K \subset \big\{x \in \mathcal{X} : K_n \ni x_n \underset{n\to\infty}{\longrightarrow} x\big\}$
$\subset \big\{x \in \mathcal{X} : x \text{ is a cluster point of } x_n \in K_n\big\} \subset K$.

As a consequence, the topology on $\mathcal{K}_+(\mathcal{X})$ generated by $d_{\rm H}$ does not depend on the choice of the metric d on $\mathcal{X}$.

We call this the *Hausdorff topology.*

The Hausdorff topology

- If $(\mathcal{X}, d)$ is separable, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.
- If $(\mathcal{X}, d)$ is complete, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.
- If $(\mathcal{X}, d)$ is compact, then so is $(\mathcal{K}_+(\mathcal{X}), d_{\rm H})$.

More generally, $\mathcal{A} \subset \mathcal{K}_+(\mathcal{X})$ is precompact iff
$\exists C \in \mathcal{K}_+(\mathcal{X})$ s.t. $K \subset C \quad \forall K \in \mathcal{A}$.

The one-point compactification

A *compactification* of a topological space $\mathcal{X}$ is a compact topological space $\overline{\mathcal{X}}$ such that $\mathcal{X}$ is a dense subset of $\overline{\mathcal{X}}$.

Let $\mathcal{X}$ be a metrisable topological space that satisfies the equivalent conditions:

(i) $\mathcal{X}$ is locally compact and separable,

(ii) $\mathcal{X}$ is an open subset of some, and hence of all of its metrisable compactifications $\overline{\mathcal{X}}$.

Let $\mathrm{Clos}(\mathcal{X})$ denote the set of closed subsets of $\mathcal{X}$.

The *one-point compactification* $\mathcal{X}_\infty := \mathcal{X} \cup \{\infty\}$ is defined by

$$\mathrm{Clos}(\mathcal{X}_\infty) := \big\{A \subset \mathcal{X}_\infty : A \cap \mathcal{X} \in \mathrm{Clos}(\mathcal{X}) \text{ and if } A \cap \mathcal{X} \text{ is not compact, then } \infty \in A\big\}.$$

Now $\mathcal{X}_\infty$ is a compact metrisable space.

The Vietoris topology

Let $\mathcal{X}$ be locally compact and separable. Then

$$\mathcal{C} := \{A \in \mathcal{K}_+(\mathcal{X}_\infty) : \infty \in A\}$$

is a closed subset of $\mathcal{K}_+(\mathcal{X}_\infty)$, and

$$\mathrm{Clos}(\mathcal{X}) \ni A \mapsto A \cup \{\infty\} \in \mathcal{C}$$

is a bijection. The topology on $\mathrm{Clos}(\mathcal{X})$ generated by

$$d_{\mathrm{V}}(A, B) := d_{\mathrm{H}}\big(A \cup \{\infty\}, A \cup \{\infty\}\big)$$

is the *Vietoris topology*.

The space $\big(\mathrm{Clos}(\mathcal{X}), d_{\mathrm{V}}\big)$ is compact and its topology does not depend on the choice of the metric d on $\mathcal{X}$.

The Hausdorff and Vietoris topologies

The Hausdorff and Vietoris topologies coincide if $\mathcal{X}$ is compact.

Both topologies can be defined on $\mathrm{Clos}(\mathcal{X})$
for any metric space $(\mathcal{X}, d)$,
but in this generality their properties are not so good.

For example, the Hausdorff topology on $\mathrm{Clos}(\mathcal{X})$
may depend on the choice of the metric d on $\mathcal{X}$
and the Vietoris topology may not be metrisable.

Best to define the Hausdorff topology only on $\mathcal{K}_+(\mathcal{X})$,

and the Vietoris topology on $\mathrm{Clos}(\mathcal{X})$
only for locally compact, separable, metrisable $\mathcal{X}$.

Squeezed space

For any metric space $\mathcal{X}$ we define the *squeezed space* $\mathcal{R}(\mathcal{X}) := (\mathcal{X} \times \mathbb{R}) \cup \{(*, -\infty), (*, \infty)\}$.

Lemma There exists a metric $d_{\rm sqz}$ on $\mathcal{R}(\mathcal{X})$ such that $d\big((x_n, t_n), (x, t)\big) \underset{n\to\infty}{\longrightarrow} 0 \quad \Leftrightarrow$

(i) $t_n \to t$ in the topology on $\overline{\mathbb{R}} := [-\infty, \infty]$,

(ii) if $t \in \mathbb{R}$, then also $x_n \to x$ in the topology on $\mathcal{X}$.

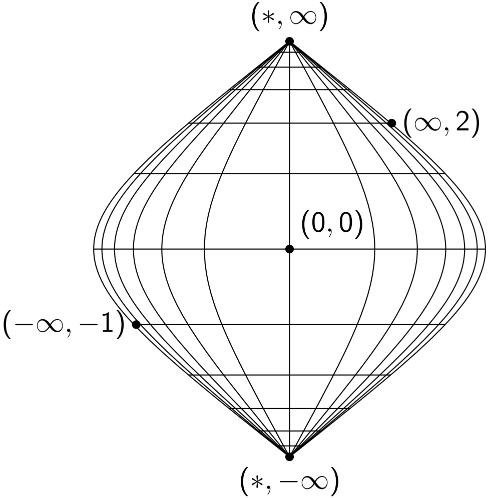
Proof Let $d_{\overline{\mathbb{R}}}$ generate the topology on $\overline{\mathbb{R}}$.
Let $\varphi : \overline{\mathbb{R}} \to [0, \infty)$ satisfy $\varphi(t) > 0 \ \Leftrightarrow \ t \in \mathbb{R}$.

Then $d_{\rm sqz}\big((x, s), (y, t)\big) :=$
$\big(\varphi(s) \wedge \varphi(t)\big)\big(d(x, y) \wedge 1\big) + \big|\varphi(s) - \varphi(t)\big| + d_{\overline{\mathbb{R}}}(s, t)$
does the trick. ∎

Idea: care less about spatial distances
when the time coordinates are large.

Squeezed space

$\mathcal{R}(\overline{\mathbb{R}})$



The split real line

The *split real line* is the set $\mathbb{R}_{\mathfrak{s}}$ consisting of all pairs $t\pm$ consisting of a real number $t \in \mathbb{R}$ and a sign $\pm \in \{-,+\}$.
For an element $\tau = t\pm$ of $\mathbb{R}_{\mathfrak{s}}$ we let $\underline{\tau} := t$ denote its real part and $\mathfrak{s}(\tau) := \pm$ its sign.
We equip $\mathbb{R}_{\mathfrak{s}}$ with the lexographic order, in which $\sigma \leq \tau$ if and only if $\underline{\sigma} < \underline{\tau}$ or $\underline{\sigma} = \underline{\tau}$ and $\mathfrak{s}(\sigma) \leq \mathfrak{s}(\tau)$.
We write $\sigma < \tau$ iff $\sigma \leq \tau$ and $\sigma \neq \tau$ and define intervals

$$\langle\!\langle \sigma, \rho \rangle\!\rangle := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau < \rho\}, \qquad [\![\sigma, \rho \rangle\!\rangle := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau < \rho\},$$
$$\langle\!\langle \sigma, \rho]\!] := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma < \tau \leq \rho\}, \qquad [\![\sigma, \rho]\!] := \{\tau \in \mathbb{R}_{\mathfrak{s}} : \sigma \leq \tau \leq \rho\}.$$

There is some redundancy, e.g., $\langle\!\langle s-, r+]\!] = [\![s+, r+]\!]$.

The split real line

We equip the split real line $\mathbb{R}_\mathfrak{s}$ with the *order topology*.
A basis for the topology is formed by all open intervals $(\!(\sigma,\rho)\!)$ with $\sigma,\rho\in\mathbb{R}_\mathfrak{s}$, $\sigma<\rho$.

(i) $\tau_n \to t+$ iff $\underline{\tau}_n \to t$ and $\tau_n \geq t+$ for n sufficiently large.

(ii) $\tau_n \to t-$ iff $\underline{\tau}_n \to t$ and $\tau_n \leq t-$ for n sufficiently large.

Lemma $\mathbb{R}_\mathfrak{s}$ is first countable, Hausdorff and separable, but not second countable and not metrisable.

Lemma For $C \subset \mathbb{R}_\mathfrak{s}^d$, the following are equivalent:

(i) C is compact,

(ii) C is sequentially compact,

(iii) C is closed and bounded.

Paths

For each closed $I \subset \mathbb{R}$ write $I_{\mathfrak{s}} := \{t\pm : t \in I\}$.

Let $\mathcal{X}$ be a metrisable topological space.
A *path* with values in $\mathcal{X}$ is a pair $\pi = (I, f)$
where $I \subset \mathbb{R}$ is a closed set and $f : I_{\mathfrak{s}} \to \mathcal{X}$ is continuous.
We write $I(\pi) := I$ and $\pi(\tau) := f(\tau)$ $\big(\tau \in I_{\mathfrak{s}}(\pi)\big)$.

We identify π with its *closed graph*

$$\pi = \big\{\big(\pi(\tau), \underline{\tau}\big) : \tau \in I_{\mathfrak{s}}(\pi)\big\} \cup \big\{(*, -\infty), (*, \infty)\big\}.$$

The set $\pi \in \Pi(\mathcal{X})$ has a natural total order defined as

$$\big(\pi(\sigma), \underline{\sigma}\big) \preceq \big(\pi(\tau), \underline{\tau}\big) \quad \Leftrightarrow \quad \sigma \leq \tau.$$

The pair $(\pi, \preceq)$ uniquely determines $I(\pi)$ and $\pi(t\pm)$ for $t \in I(\pi)$.

We let $\Pi(\mathcal{X})$ denote the set of paths with values in $\mathcal{X}$.

Paths

The total order $\preceq$ on π has nice properties:

(i) $(x,s) \preceq (y,t)$ for all $s < t$,

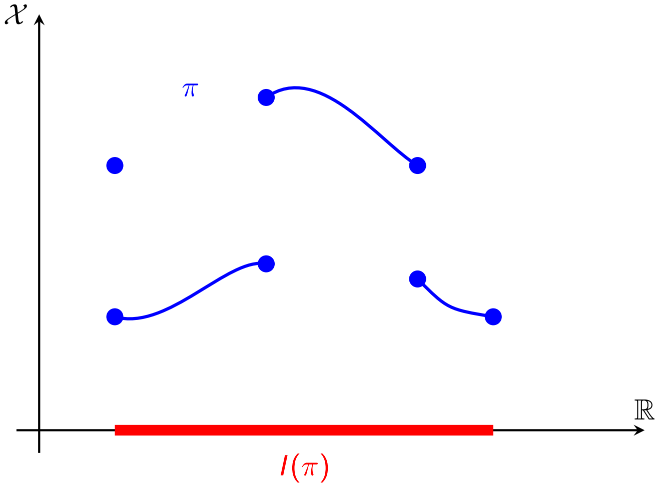
(ii) $\{(z,z') : z \preceq z'\}$ is a closed subset of π^2.

Lemma There is a one-to-one correspondence between paths and totally ordered compact subsets $(\pi, \preceq)$ of $\mathcal{R}(\mathcal{X})$ that satisfy (i) and (ii) as well as

(iii) $(*, \pm\infty) \in \pi$,

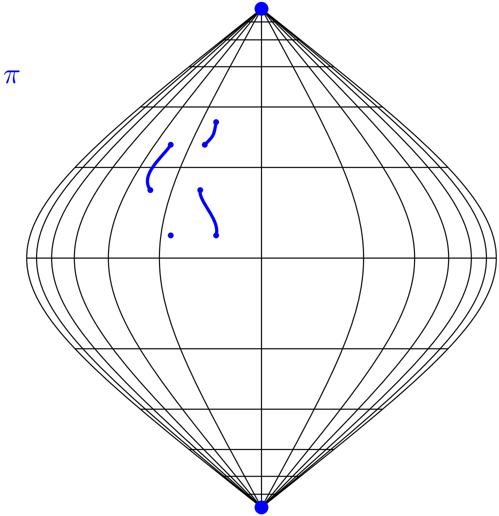
(iv) $\big|\big\{x \in \mathcal{X} : (x,t) \in \pi\big\}\big| \leq 2 \quad \forall t \in \mathbb{R}$.

Paths



Paths more or less correspond to *cadlag* functions.

Paths



The closed graph as a subset of $\mathcal{R}(\mathcal{X})$.

Paths

We set $\Pi^{|}(\mathcal{X}) := \{\pi \in \Pi : I \text{ is an interval}\}$.
For $\pi \in \Pi^{|}(\mathcal{X})$ we set $\sigma_\pi := \inf I(\pi)$ and $\tau_\pi := \sup I(\pi)$.

Note that a path π can jump at its starting time σ_π.

We set[1]

$$\begin{aligned}
\Pi_{\rm c}(\mathcal{X}) &:= \{\pi \in \Pi(\mathcal{X}) : \pi(t-) = \pi(t+)\ \forall t \in I(\pi)\},\\
\Pi^{\uparrow}(\mathcal{X}) &:= \{\pi \in \Pi^{|}(\mathcal{X}) : \tau_\pi = \infty\},\\
\Pi^{\downarrow}(\mathcal{X}) &:= \{\pi \in \Pi^{|}(\mathcal{X}) : \sigma_\pi = -\infty\}.
\end{aligned}$$

Then $\Pi^{\uparrow}_{\rm c}(\overline{\mathbb{R}}) := \Pi_{\rm c}(\overline{\mathbb{R}}) \cap \Pi^{\uparrow}(\overline{\mathbb{R}})$ is the classical path space introduced by Fontes, Isopi, Newman, and Ravishankar (AoP 2004).

[1] For $I(\pi) = \emptyset$ we use the conventions $\sigma_\pi := -\infty$, $\tau_\pi := \infty$.

Convergence of paths

We equip $\Pi_{\rm c}(\mathcal{X})$ with the metric:

$$d_{\rm H}(\pi_1,\pi_2) = \inf_{R\in{\rm Cor}(\pi_1,\pi_2)} \sup_{((x_1,t_1),(x_2,t_2))\in R} d_{\rm sqz}\big((x_1,t_1),(x_2,t_2)\big).$$

This corresponds to *locally uniform convergence*.

The topology on $\Pi_{\rm c}(\mathcal{X})$ does not depend on the choice of the metric d on $\mathcal{X}$.

The topology on $\Pi^{\uparrow}_{\rm c}(\overline{\mathbb{R}})$ corresponds to the one introduced by Fontes, Isopi, Newman, and Ravishankar (AoP 2004).

Convergence of paths

Recall that $\preceq$ denotes the natural total order on π, defined as

$$(\pi(\sigma),\underline{\sigma}) \preceq (\pi(\tau),\underline{\tau}) \quad \Leftrightarrow \quad \sigma \leq \tau.$$

For totally ordered compact sets K_1, K_2, let $\mathrm{Cor}_+(K_1, K_2)$ denote the set of correspondences $R \in \mathrm{Cor}(K_1, K_2)$ that are *monotone* in the sense that:

$$\nexists\ (x_1, x_2), (y_1, y_2) \in R \text{ such that } x_1 \prec y_1 \text{ and } y_2 \prec x_2,$$

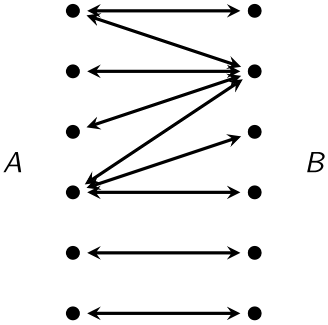
where $x \prec y$ means $x \preceq y$ and $x \neq y$.

We equip $\Pi(\mathcal{X})$ with the metric:

$$d_{\mathrm{J1}}(\pi_1, \pi_2) = \inf_{R \in \mathrm{Cor}_+(\pi_1,\pi_2)} \sup_{((x_1,t_1),(x_2,t_2)) \in R} d_{\mathrm{sqz}}\big((x_1, t_1), (x_2, t_2)\big).$$

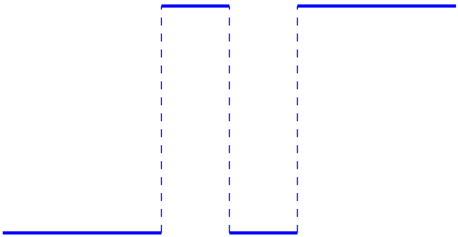
This corresponds to convergence in *Skorohod's J1 topology*.

Convergence of paths



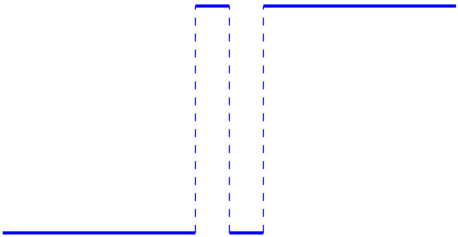
A monotone correspondence between two totally ordered sets A and B.

Convergence of paths



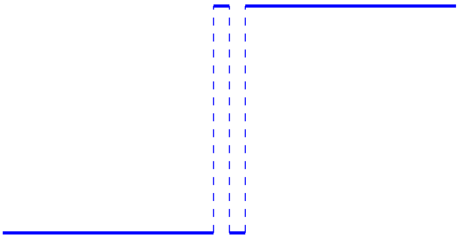
A sequence that converges in d_{H} but not in d_{J1}.

Convergence of paths



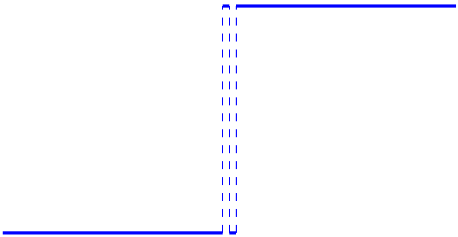
A sequence that converges in d_{H} but not in d_{J1}.

Convergence of paths



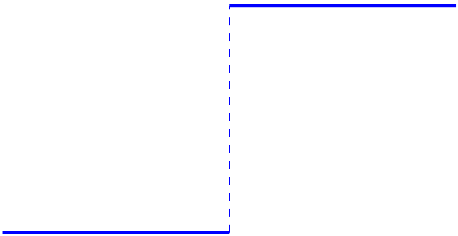
A sequence that converges in d_{H} but not in d_{J1}.

Convergence of paths



A sequence that converges in d_{H} but not in d_{J1}.

Convergence of paths



A sequence that converges in d_{H} but not in d_{J1}.

Convergence of paths

- The topology on $\Pi(\mathcal{X})$ does not depend on the choice of the metric d on $\mathcal{X}$.
- The topology on $\Pi_{\rm c}(\mathcal{X})$ is the induced topology from $\Pi(\mathcal{X})$.

Recall that a topological space is *Polish* if it is separable and there exists a complete metric generating the topology.

- If $\mathcal{X}$ is separable, then so are $\Pi_{\rm c}(\mathcal{X})$ and $\Pi(\mathcal{X})$.
- If $\mathcal{X}$ is Polish, then so are $\Pi_{\rm c}(\mathcal{X})$ and $\Pi(\mathcal{X})$.

However, $d_{\rm H}$ and $d_{\rm J1}$ are typically not complete even when d is.

- $\Pi^{|}_{\rm c}(\mathcal{X})$ is a closed subspace of $\Pi(\mathcal{X})$.

The M1 topology

Set $\langle s, t \rangle := [s \wedge t, s \vee t] \quad (s, t \in \overline{\mathbb{R}})$.

The *filled graph* of $\pi \in \Pi(\overline{\mathbb{R}})$ is

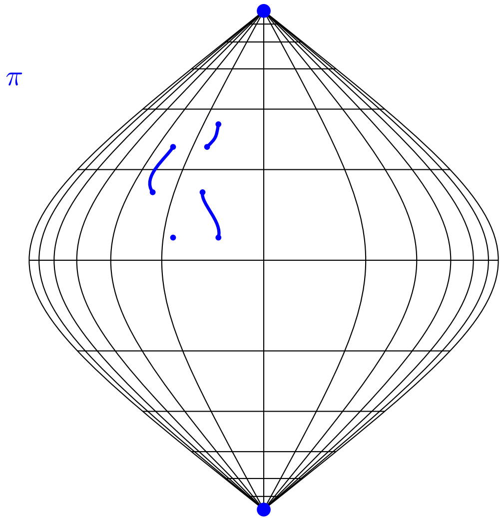
$$\overline{\pi} := \big\{(x, t) : t \in I(\pi) : x \in \langle \pi(t-), \pi(t+) \rangle \big\} \cup \big\{(-\infty, *), (\infty, *)\big\}.$$

Write $(x_1, t_1) \preceq (x_2, t_2)$ if $t_1 < t_2$ or $t_1 = t_2 =: t$ and x_2 is closer to $\pi(t+)$ than x_1. The M1 topology on $\Pi(\overline{\mathbb{R}})$ is generated by

$$d_{\mathrm{J1}}(\pi_1, \pi_2) = \inf_{R \in \mathrm{Cor}_+(\overline{\pi}_1, \overline{\pi}_2)} \sup_{(z_1, z_2) \in R} d_{\mathrm{sqz}}(z_1, z_2).$$

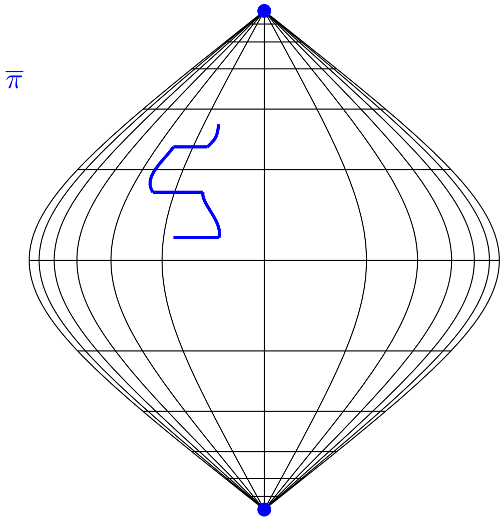
This corresponds to convergence in *Skorohod's M1 topology*.

The M1 topology



The closed graph π.

The M1 topology



The filled graph $\overline{\pi}$.

Pathological paths

Set $\Pi := \Pi(\overline{\mathbb{R}})$.

$$\Delta^2_{T,\delta}(\pi) := \big\{ (x_1, x_2) : (x_1, t_1) \preceq (x_2, t_2) \\ (x_i, t_i) \in \pi,\ \ t_i \in [-T, T],\ \ t_2 - t_1 \leq \delta \big\},$$

$$\Delta^3_{T,\delta}(\pi) := \big\{ (x_1, x_2, x_3) : (x_1, t_1) \preceq (x_2, t_2) \preceq (x_3, t_3) \\ (x_i, t_i) \in \pi,\ \ t_i \in [-T, T],\ \ t_3 - t_1 \leq \delta \big\}.$$

$$\Pi^+_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \in \Delta^2_{T,\delta}(\pi) \text{ s.t. } x_1 \leq r,\ \ r + \varepsilon \leq x_2 \big\},$$

$$\Pi^-_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \in \Delta^2_{T,\delta}(\pi) \text{ s.t. } x_2 \leq r,\ \ r + \varepsilon \leq x_1 \big\},$$

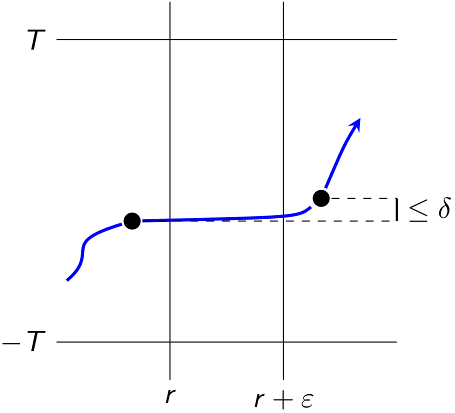
$$\Pi^{+-}_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \in \Delta^3 \text{ s.t. } x_1, x_3 \leq r,\ \ r + \varepsilon \leq x_2 \big\},$$

$$\Pi^{-+}_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \in \Delta^3 \text{ s.t. } x_2 \leq r,\ \ r + \varepsilon \leq x_1, x_3 \big\},$$

$$\Pi^{++}_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \text{ s.t. } x_1 \leq r,\ \ r + \varepsilon \leq x_2 \leq r + 2\varepsilon,\ \ r + 3\varepsilon \leq x_3 \big\},$$

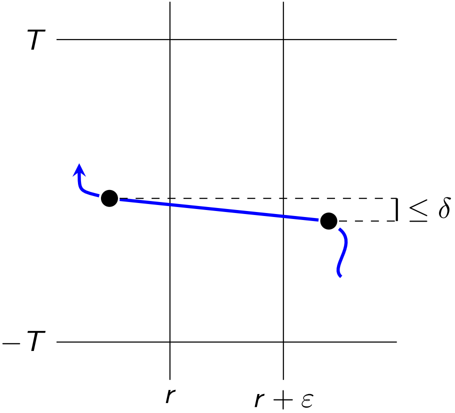
$$\Pi^{--}_{T,\delta,\varepsilon,r} := \big\{\pi : \exists \vec{x} \text{ s.t. } x_3 \leq r,\ \ r + \varepsilon \leq x_2 \leq r + 2\varepsilon,\ \ r + 3\varepsilon \leq x_1 \big\}.$$

Pathological paths



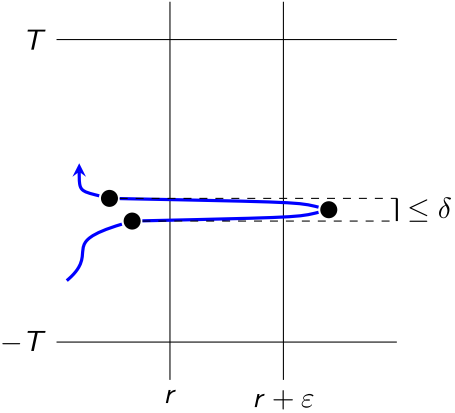
A path $\pi \in \Pi^{+}_{T,\delta,\varepsilon,r}$.

Pathological paths



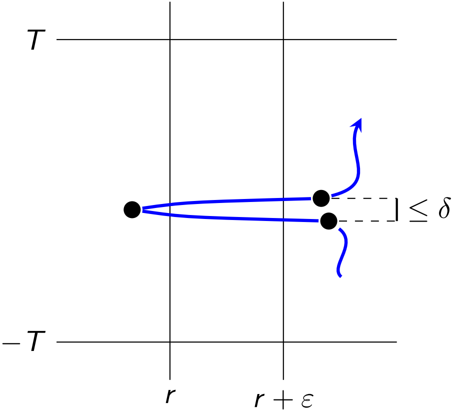
A path $\pi \in \Pi^{-}_{T,\delta,\varepsilon,r}$.

Pathological paths



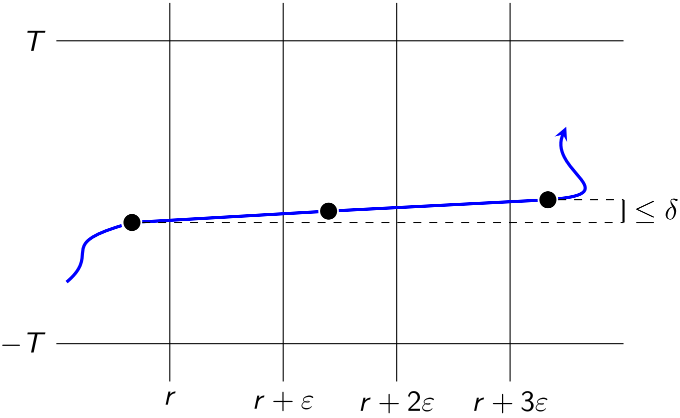
A path $\pi \in \Pi^{+-}_{T,\delta,\varepsilon,r}$.

Pathological paths



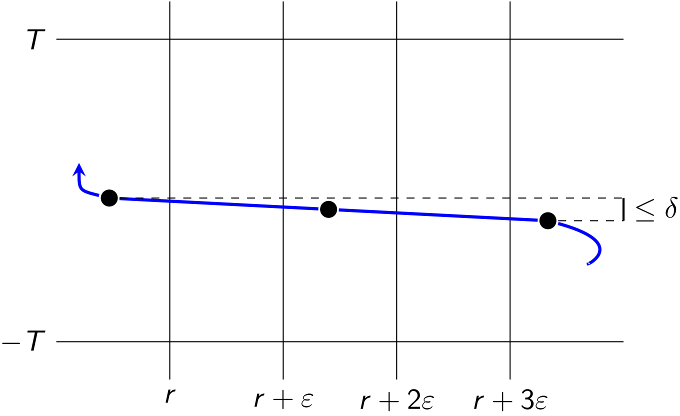
A path $\pi \in \Pi^{-+}_{T,\delta,\varepsilon,r}$.

Pathological paths



A path $\pi \in \Pi^{++}_{T,\delta,\varepsilon,r}$.

Pathological paths



A path $\pi \in \Pi^{--}_{T,\delta,\varepsilon,r}$.

Random sets of paths

We equip $\mathcal{K}_+(\Pi_\mathrm{c})$ and $\mathcal{K}_+(\Pi)$ with the Hausdorff topology, under which they are Polish.

Theorem A sequence $\mathcal{A}_n$ of random variables with values in $\mathcal{K}_+(\Pi_\mathrm{c})$ is tight if and only if

$$\lim_{\delta\to 0}\sup_n \mathbb{P}\big[\Pi^2_{T,\delta,\varepsilon,r}\cap\mathcal{A}_n\neq\emptyset\big]=0\quad\forall T,\varepsilon,r,$$

where $\Pi^2_{T,\delta,\varepsilon,r}:=\Pi^+_{T,\delta,\varepsilon,r}\cup\Pi^-_{T,\delta,\varepsilon,r}$.

Random sets of paths

Theorem A sequence $\mathcal{A}_n$ of random variables with values in $\mathcal{K}_+(\Pi)$ is tight with respect to the J1 or M1 topologies if and only if

$$\lim_{\delta\to 0}\sup_n \mathbb{P}\big[\Pi^{\mathrm{X}}_{T,\delta,\varepsilon,r}\cap\mathcal{A}_n\neq\emptyset\big]=0\quad\forall T,\varepsilon,r,$$

where for $\mathrm{X}=\mathrm{J1}$ or $\mathrm{M1}$,

$$\begin{aligned}\Pi^{\mathrm{J1}}_{T,\delta,\varepsilon,r}&:=\Pi^{++}_{T,\delta,\varepsilon,r}\cup\Pi^{+-}_{T,\delta,\varepsilon,r}\cup\Pi^{-+}_{T,\delta,\varepsilon,r}\cup\Pi^{--}_{T,\delta,\varepsilon,r},\\ \Pi^{\mathrm{M1}}_{T,\delta,\varepsilon,r}&:=\Pi^{+-}_{T,\delta,\varepsilon,r}\cup\Pi^{-+}_{T,\delta,\varepsilon,r}.\end{aligned}$$

Noncrossing sets of paths

For $\pi_1, \pi_2 \in \Pi^{|}$ write $\pi_1 \lhd \pi_2$ if π_1, π_2 can be extended to paths $\pi_1', \pi_2' \in \Pi^{\updownarrow}$ such that $\pi_1'(t\pm) \leq \pi_2'(t\pm)$ for all $t \in \mathbb{R}$.

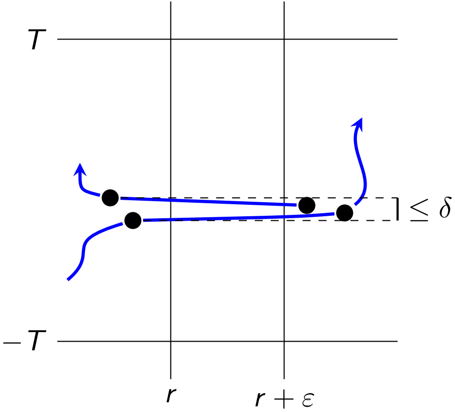
Let $\mathcal{K}_{\rm nc}(\Pi^{|}) := \big\{\mathcal{A} \in \mathcal{K}_+(\Pi^{|}) : \pi_1 \lhd \pi_2 \text{ or } \pi_2 \lhd \pi_1 \ \forall \pi_1, \pi_2 \in \mathcal{A}\big\}$.

$$\Gamma^{\rm M1}_{T,\delta,\varepsilon,r} := \big\{ (\pi_1, \pi_2) \in \Pi^2 : \exists (x_i, y_i) \in \Delta^2_{T,\delta}(\pi_i), \\ \text{s.t. } (t_1 \vee t_2) - (s_1 \wedge s_2) \leq \delta,\ x_1, y_2 \leq r,\ r + \varepsilon \leq y_1, x_2 \big\}.$$

Theorem A sequence $\mathcal{A}_n$ of random variables with values in $\mathcal{K}_{\rm nc}(\Pi^{|})$ is tight with respect to the M1 topology if and only if

$$\lim_{\delta \to 0} \sup_n \mathbb{P}\big[\Gamma^{\rm M1}_{T,\delta,\varepsilon,r} \cap (\mathcal{A}_n \times \mathcal{A}_n) \neq \emptyset\big] = 0 \quad \forall T, \varepsilon, r.$$

Noncrossing sets of paths



A pair of paths $(\pi_1, \pi_2) \in \Gamma^{\mathrm{M1}}_{T,\delta,\varepsilon,r}$.