Universality of the Brownian net

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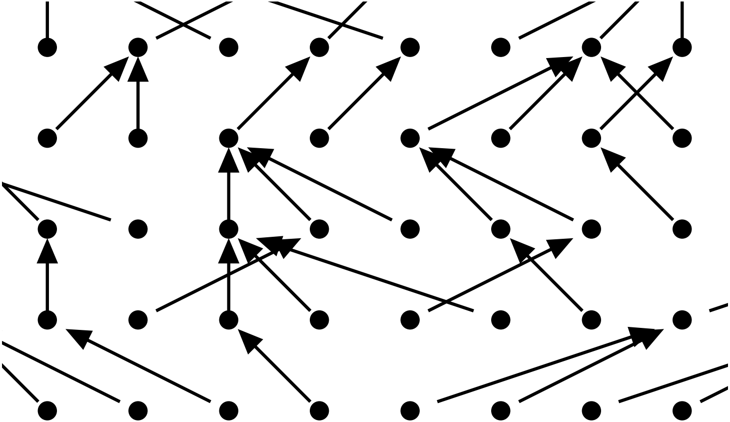
Eurandom, April 17, 2024

A discrete web

0	2	-2	1	-3	2	1	0
1	0	1	1	2	1	1	-1
-1	-3	0	-1	-2	-1	-2	-1
0	2	0	-1	2	-3	-1	3
-1	-2	-2	-1	3	2	3	2

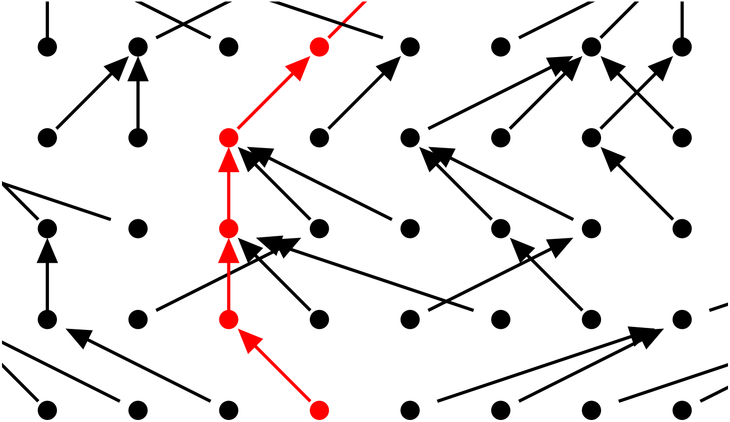
Let $\big(\omega(x,t)\big)_{(x,t)\in\mathbb{Z}^2}$ be i.i.d. $\mathbb{Z}$-valued random variables with common law a.

A discrete web



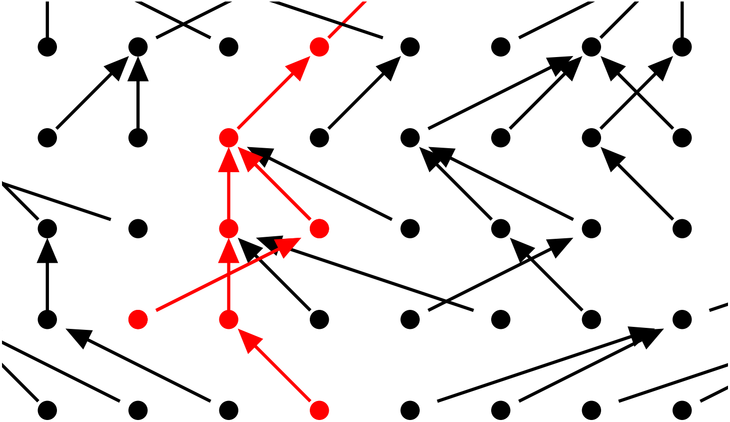
For each $(x,t) \in \mathbb{Z}^2$, draw an arrow from (x,t) to $\big(x+\omega(x,t), t+1\big)$.

A discrete web



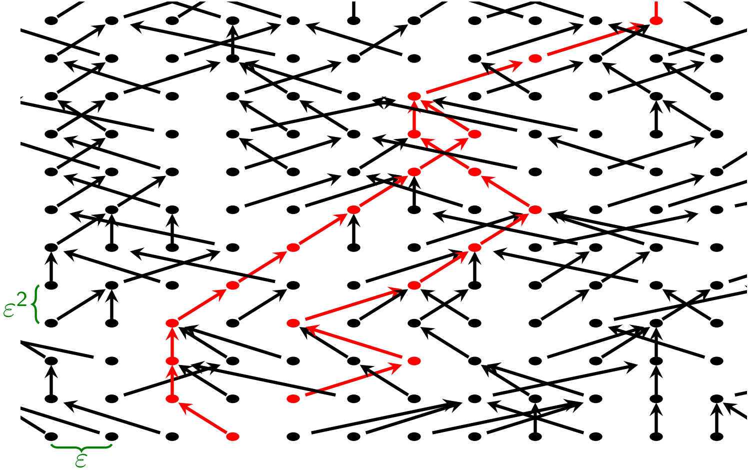
At each (x, t) there starts a unique path
that is a random walk with increment law a.

A discrete web



Paths evolve independently until they *coalesce*.

A discrete web



We rescale space by ε
and time by ε^2.

The diffusive scaling limit

We are interested in the diffusive scaling limit of the collection of all *open paths*, i.e., paths that follow the arrows.

We assume that

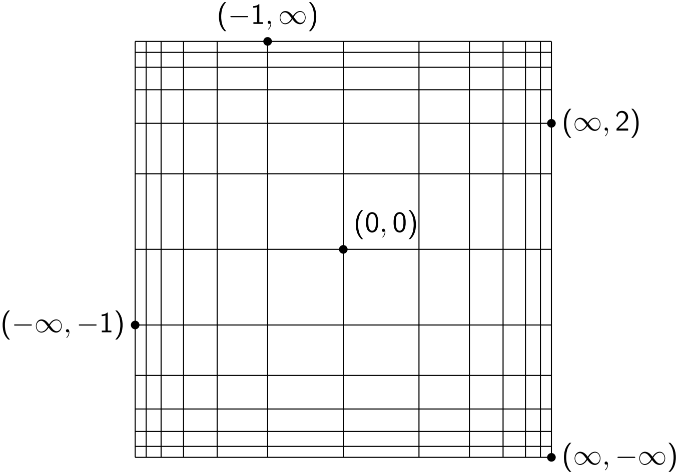
$$\sum_{k\in\mathbb{Z}} a(k)k = 0 \quad \text{and} \quad \sigma^2 := \sum_{k\in\mathbb{Z}} a(k)k^2 < \infty.$$

Then the diffusive scaling limit of a single path is Brownian motion with diffusion rate σ^2.

If moreover a is irreducible and aperiodic, then in the diffusive scaling limit, two Brownian paths *coalesce* as soon as they meet.

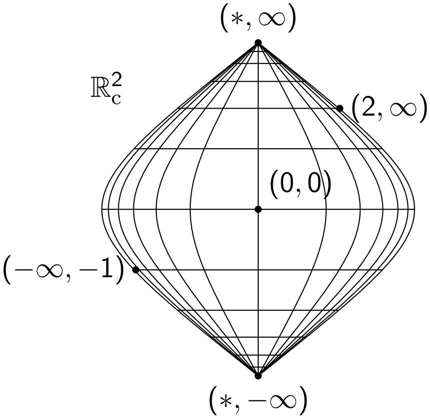
Aim Describe the scaling limit of the set of all paths.

Topological matters



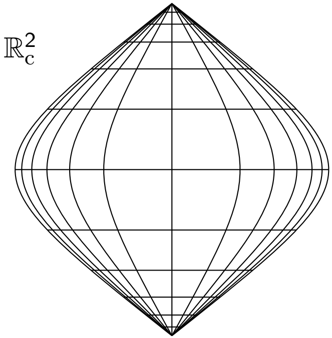
We first compactify $\mathbb{R}^2$ to $\overline{\mathbb{R}}^2 = [-\infty, \infty]^2 \ldots$

Topological matters



$\ldots$ and then contract $[-\infty,\infty]\times\{-\infty\}$
and $[-\infty,\infty]\times\{\infty\}$ to single points.

Topological matters

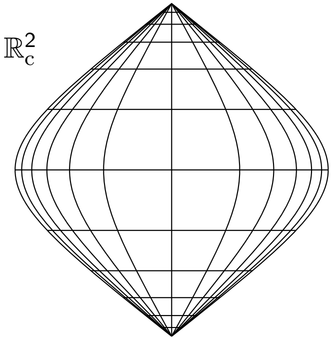


Alternatively, map $\mathbb{R}^2$ into itself with the map

$$\Theta(x,t) := \Big(\frac{\tanh(x)}{1+|t|}, \tanh(t)\Big),$$

and take the closure.

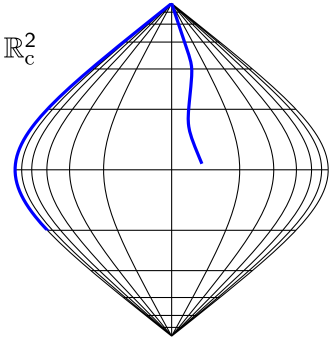
Topological matters



Another equivalent formulation is: take the completion of $\mathbb{R}^2$ w.r.t. the metric

$$d(z,z') := \big|\Theta(z) - \Theta(z')\big|.$$

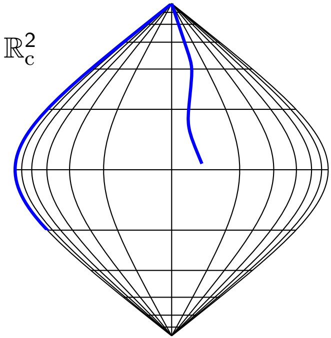
Topological matters



A *path* is a compact set $\pi \subset \mathbb{R}^2_c$ such that:

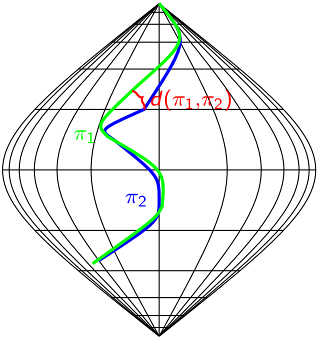
- $\big\{t : \exists x \text{ s.t. } (x,t) \in \pi\big\} = [\sigma_\pi, \infty]$ for some $-\infty \leq \sigma_\pi \leq \infty$,
- $\forall t \in [\sigma_\pi, \infty],\ \exists!\ \pi(t) \text{ s.t. } \big(\pi(t), t\big) \in \pi$.

Topological matters



For each path π, the function $[\sigma_\pi, \infty) \ni t \mapsto \pi(t)$ is continuous and $\pi \backslash \{(*, \infty)\}$ is the graph of this function.

Topological matters



We equip the space Π of all paths with the Hausdorff metric

$$d_{\mathrm{H}}(\pi_1,\pi_2) = \sup_{z_1\in\pi_1} \inf_{z_2\in\pi_2} d(z_1,z_2) \vee \sup_{z_2\in\pi_2} \inf_{z_1\in\pi_1} d(z_1,z_2).$$

Coalescing Brownian motions

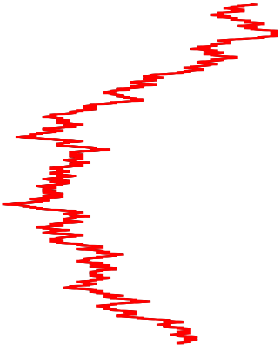
Let $(z_i)_{i\geq 1} = (x_i, t_i)_{i\geq 1}$ be points in $\mathbb{R}^2$.
Let $(B_i)_{i\geq 1}$ with $B_i = \big(B_i(t)\big)_{t\geq t_i}$ be independent Brownian motions started from $B_i(t_i) = x_i$.

Define inductively $\tau_i := \inf\{t \geq t_i : \big(B_i(t), t\big) \in \bigcup_{k=1}^{i-1} A_k\}$
with $A_i := \big\{\big(B_i(t), t\big) : t_i \leq t < \tau_i\big\}$.
For $i \geq 2$ define $\kappa(i) < i$ by $\big(B_i(\tau_i), \tau_i\big) \in A_{\kappa(i)}$.

Then we can inductively define
coalescing Brownian motions $(P_i)_{i\geq 1}$ *started from* $(z_i)_{i\geq 1}$ by:

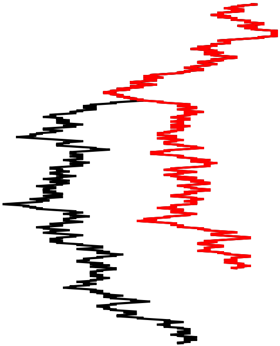
$P_i(t) := B_i(t)\ (t_i \leq t < \tau_i)$
$P_i(t) := P_{\kappa(i)}(t)\ (\tau_i \leq t < \infty)$

Coalescing Brownian motions



Coalescing Brownian motions.

Coalescing Brownian motions



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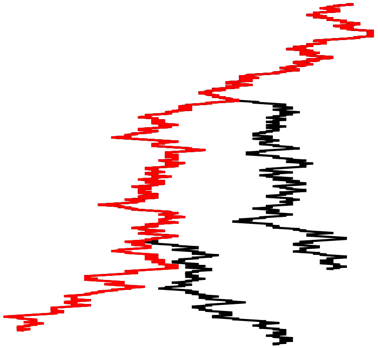

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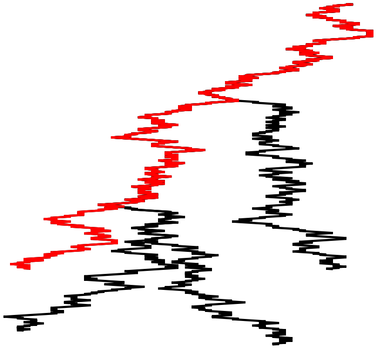

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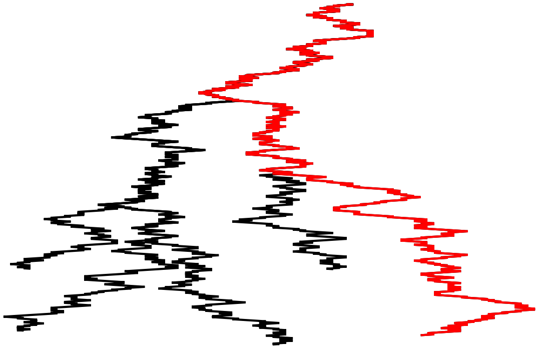

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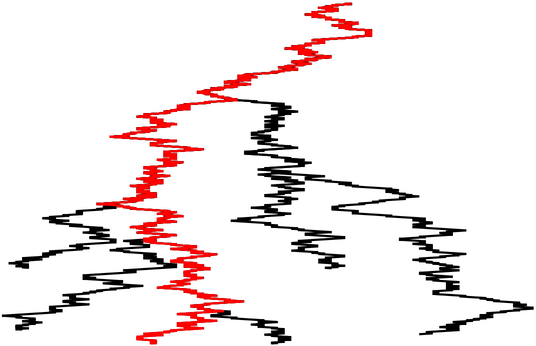
Coalescing Brownian motions



Coalescing Brownian motions.

The Brownian web

[Fontes, Isopi, Newman & Ravishankar (AoP 2004)]
Let $(P_i)_{i\geq 1}$ be coalescing Brownian motions started from $(z_i)_{i\geq 1}$.
Assume that $\{z_i : i \in \mathbb{N}_+\}$ is dense in $\mathbb{R}^2$.
Then $\{P_i : i \in \mathbb{N}_+\} \subset \Pi$ is precompact and the law of

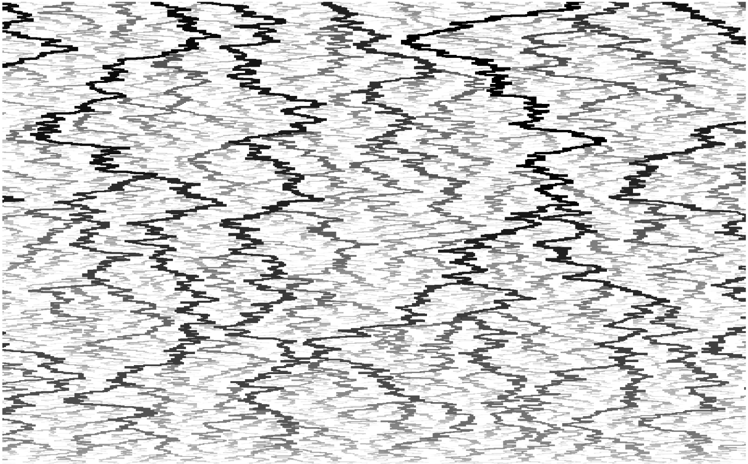
$$\mathcal{W} := \overline{\{P_i : i \in \mathbb{N}_+\}}$$

does not depend on $\{z_i : i \in \mathbb{N}_+\}$.

The random compact set $\mathcal{W}$ is called the *Brownian web*.

We let $\mathcal{W}^\sigma$ denote the Brownian web with diffusion rate σ^2.

The Brownian web



Artist's impression of the Brownian web.

Discrete webs

Recall that $\big(\omega(x,t)\big)_{(x,t)\in\mathbb{Z}^2}$ are i.i.d. $\mathbb{Z}$-valued with common law a.

Let $\mathcal{U}$ be the space of paths π such that

- $\sigma_\pi \in \overline{\mathbb{Z}} := \mathbb{Z} \cup \{\pm\infty\}$,

and either

- $\pi(t) \in \mathbb{Z}$ for $\sigma_\pi \leq t \in \mathbb{Z}$ with linear interpolation,
- $\pi(t+1) = \pi(t) + \omega\big(\pi(t), t\big) \quad (\sigma_\pi \leq t \in \mathbb{Z})$,

or $\pi(t) = -\infty$ for all $\sigma_\pi \leq t < \infty$,
or $\pi(t) = +\infty$ for all $\sigma_\pi \leq t < \infty$.

Lemma $\mathcal{U}$ is a.s. a compact subset of Π.

Scaling limit of discrete webs

The discrete web $\mathcal{U}$ and the Brownian web $\mathcal{W}$
are a.s. compact subsets of Π.

We equip the space $\mathcal{K}(\Pi)$ of compact subsets of Π with the Hausdorff metric

$$d_{\mathrm{H}}(\mathcal{U}_1,\mathcal{U}_2) = \sup_{\pi_1\in\mathcal{U}_1} \inf_{\pi_2\in\mathcal{U}_2} d_{\mathrm{H}}(\pi_1,\pi_2) \vee \sup_{\pi_2\in\mathcal{U}_2} \inf_{\pi_1\in\mathcal{U}_1} d_{\mathrm{H}}(\pi_1,\pi_2).$$

We define diffusive scaling maps $S_\varepsilon : \mathbb{R}^2_{\mathrm{c}} \to \mathbb{R}^2_{\mathrm{c}}$ by

$$S_\varepsilon(x,t) := (\varepsilon x, \varepsilon^2 t),$$

and set

$$S_\varepsilon(\pi) := \big\{S_\varepsilon(x,t) : (x,t) \in \pi\big\} \quad \text{and} \quad S_\varepsilon(\mathcal{U}) := \big\{S_\varepsilon(\pi) : \pi \in \mathcal{U}\big\}.$$

Scaling limit of discrete webs

Theorem [Belhaouari, Mountford, Sun, & Valle '06]
Assume that $\sum_{k\in\mathbb{Z}} a(k)|k|^{3+\delta} < \infty$ for some $\delta > 0$. Then

$$\mathbb{P}\big[S_\varepsilon(\mathcal{U}) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{W}^\sigma \in \cdot\,\big],$$

where $\Rightarrow$ denotes weak convergence on $\mathcal{K}(\Pi)$.

[Newman, Ravishankar, & Sun '05] proved convergence if

$$\sum_{k\in\mathbb{Z}} a(k)|k|^5 < \infty.$$

[Fontes, Isopi, Newman & Ravishankar '04] proved it for

$$a(-1) = \tfrac{1}{2} = a(1).$$

The third moment condition

The third moment condition is sharp. Assume that $a(k) = a(-k)$ and set

$$A(n) := \sum_{k=n}^{\infty} a(k) \qquad (n \geq 0).$$

Then

$$\sup\Big\{\beta \geq 0 : \sum_{k \in \mathbb{Z}} a(k)|k|^{\beta} < \infty\Big\} = \sup\Big\{\beta \geq 0 : \lim_{n\to\infty} A(n)n^{\beta} = 0\Big\}.$$

If $\limsup_{n\to\infty} A(n)n^3 > 0$, then in the diffusive scaling limit there are macroscopic jumps all over the place.

Macroscopic jumps

Define $T_\varepsilon(x,y,t) := (\varepsilon x, \varepsilon y, \varepsilon^2 t)$ and let

$$J := \big\{(x,y,t) : (x,t) \in \mathbb{Z}^2,\ y = x + \omega(x,t)\big\}.$$

Assume that $c := \lim_{n\to\infty} A(n)n^3 > 0$. Then

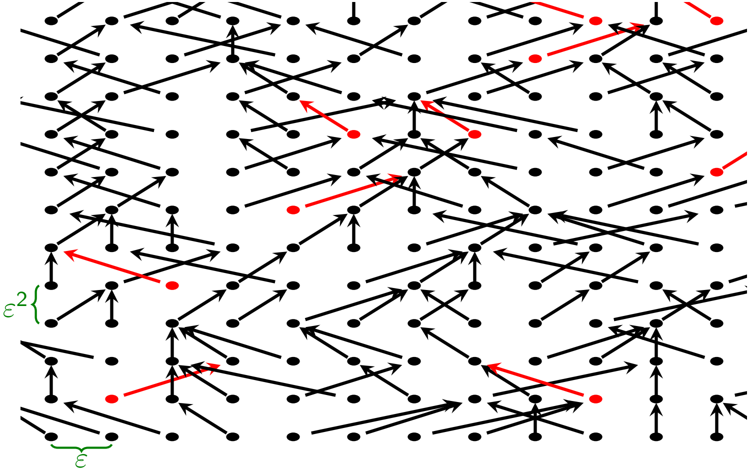
$$\mathbb{P}\big[T_\varepsilon(J) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\Xi^c \in \cdot\,\big],$$

where Ξ^c is a Poisson point set on $\{(x,y,t) \in \mathbb{R}^3 : x \neq y\}$ with intensity measure

$$\mu\big(\mathrm{d}(x,y,t)\big) := c|x-y|^{-4}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

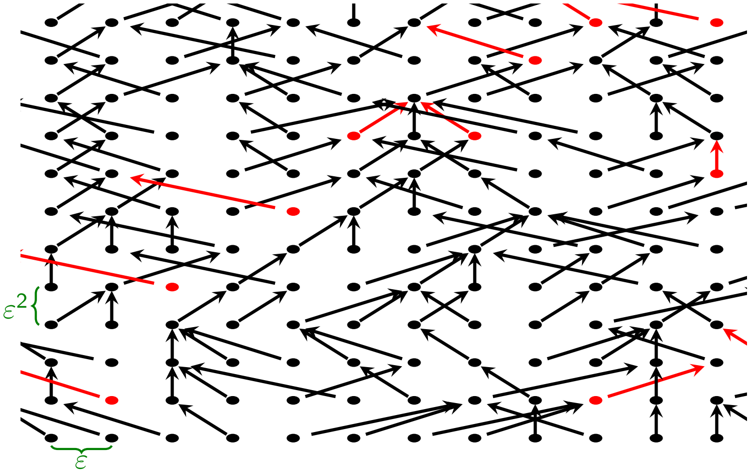
[Berestycki, Garban, & Sen '15] have shown that if you use a topology that ignores the piece of a path near its starting time, then for convergence to $\mathcal{W}^\sigma$ it suffices if $\sum_{k\in\mathbb{Z}} a(k)k^2 < \infty$.

Modified webs



We select an ε-fraction of our arrows.

Modified webs



And resample them.

Modified webs

In the diffusive scaling limit, we obtain two coupled Brownian webs.

Following **[Howitt and Warren '09]**, a pair of *sticky Brownian motions with parameter* θ is a pair $(B^1_t, B^2_t)_{t\geq 0}$ of standard Brownian motions adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that

$$\begin{aligned} &\text{(i)} \quad B^1_t B^2_t - \int_0^t 1_{\{B^1_s = B^2_s\}} \mathrm{d}s, \\ &\text{(ii)} \quad |B^1_t - B^2_t| - 2\theta \int_0^t 1_{\{B^1_s = B^2_s\}} \mathrm{d}s, \end{aligned}$$

are martingales w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

Sticky Brownian motions spend positive Lebesgue time together.

Sticky webs

Following **[Howitt and Warren '09]**, a pair of *sticky Brownian webs with parameter* θ is a pair $(\mathcal{W}^1, \mathcal{W}^2)$ of standard Brownian webs such that for each $x_1, x_2, s \in \mathbb{R}$, the a.s. unique paths $\pi^i \in \mathcal{W}^i$ started at (x_i, s) $(i = 1, 2)$ form a pair of Brownian motions with parameter θ that is adapted to the natural filtration generated by the Brownian webs $(\mathcal{W}^1, \mathcal{W}^2)$.

It can be shown that this determines the joint law of $(\mathcal{W}^1, \mathcal{W}^2)$ uniquely.

Convergence to sticky webs

Theorem Let $(\mathcal{U}^1_\varepsilon, \mathcal{U}^2_\varepsilon)$ be a pair of sticky discrete webs and let S^σ_ε denote the scaling map

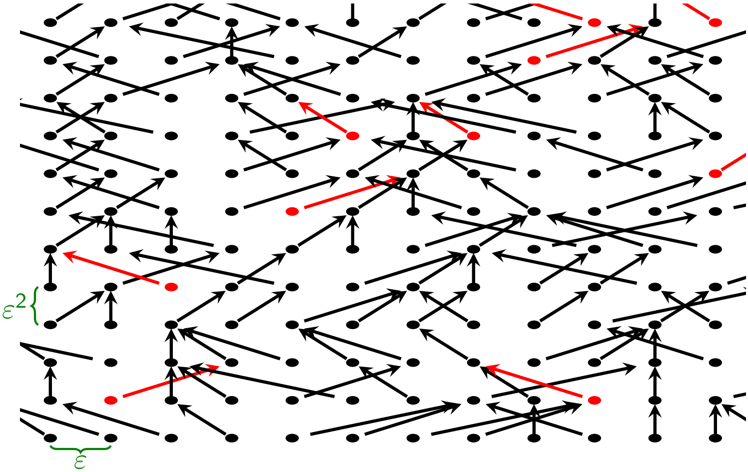
$$S^\sigma_\varepsilon(x,t) := (\varepsilon x, \sigma^2\varepsilon^2 t).$$

Then

$$\mathbb{P}\big[S^\sigma_\varepsilon(\mathcal{U}^1_\varepsilon, \mathcal{U}^2_\varepsilon) \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[(\mathcal{W}^1, \mathcal{W}^2) \in \cdot\,\big],$$

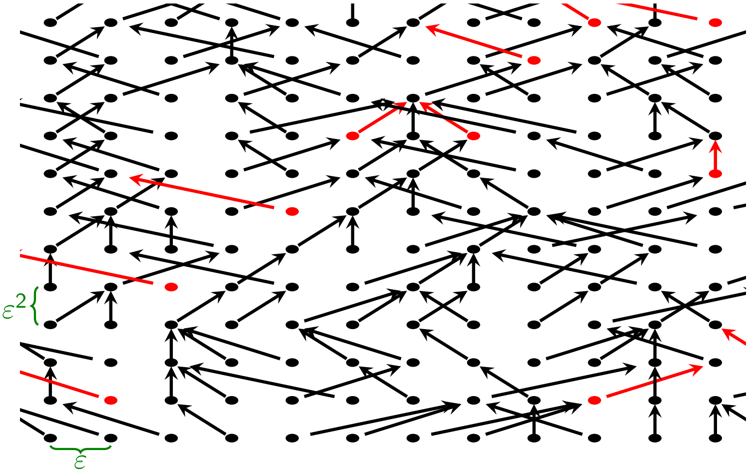
where $(\mathcal{W}^1, \mathcal{W}^2)$ is a pair of sticky Brownian webs with parameter 1.

Discrete nets



If we overlay the two modified discrete webs. . .

Discrete nets



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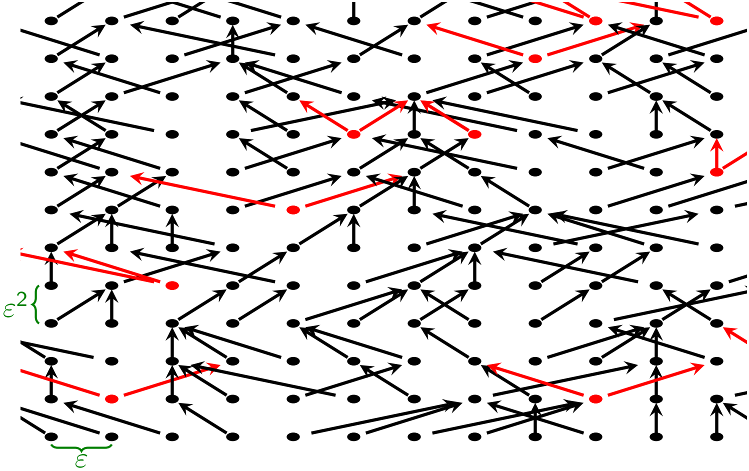
Discrete nets



Then we obtain a discrete *net*.

Discrete nets

The discrete net $\mathcal{V}_\varepsilon$ is the collection of paths π such that

- $\pi(t+1) - \pi(t) \in \{\omega^1(\pi(t), t), \omega^2(\pi(t), t)\} \quad (\sigma_\pi \leq t \in \mathbb{Z})$.

[Sun, S. & Yu '24] Assume that $\sum_{k \in \mathbb{Z}} a(k)|k|^{3+\delta} < \infty$ for some $\delta > 0$. Then

$$\mathbb{P}\big[S_\varepsilon^\sigma(\mathcal{V}_\varepsilon) \in \cdot\,\big] \underset{\varepsilon \to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{N} \in \cdot\,\big],$$

where $\Rightarrow$ denotes weak convergence on $\mathcal{K}(\Pi)$ and $\mathcal{N}$ is a random compact set of paths called the *Brownian net*.

[Sun & S. '08] proved convergence if

$$a(-1) = \tfrac{1}{2} = a(1).$$

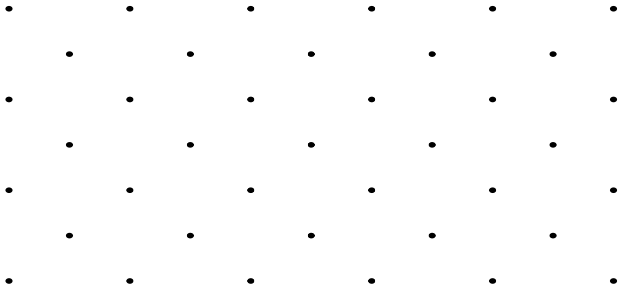
For the Brownian web, analogue results were proved in '04 and '06.

What took us so long?

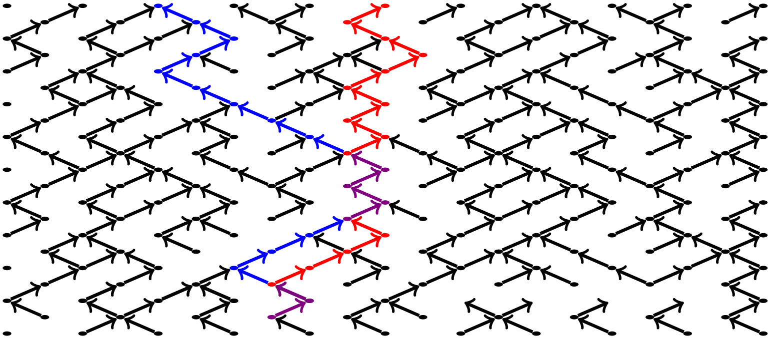
The nearest-neighbour case

The random walk kernel $a(-1) = \frac{1}{2} = a(1)$ is periodic.
This forces us to restrict ourselves to the even sublattice

$$\mathbb{Z}^2_{\mathrm{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even}\}$$



Left- and right-most paths



With probability $\frac{1}{2}\varepsilon$ there are two arrows.
We can define left- and right-most paths that always choose the left of right arrow, if there is a choice.

Left- and right-most paths

In the limit, left- and right-most paths interact with a form of sticky interaction:

$$\mathrm{d}L_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t - \mathrm{d}t,$$
$$\mathrm{d}R_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \mathrm{d}t,$$

where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t are subject to the constraint that $L_t \leq R_t$ for all $t \geq \tau := \inf\{u \geq 0 : L_u = R_u\}$.

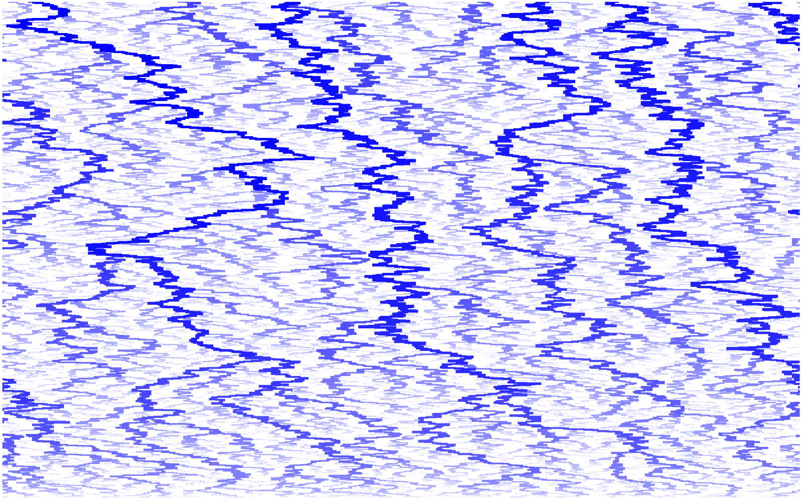
Left- and right-most paths

Alternatively, we can describe the joint evolution of $(L_t, R_t)_{t\geq 0}$ as before by a martingale problem. In this case, $(L_t)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ are Brownian motions with drift -1 and $+1$, respectively, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that

$$\text{(i)} \quad L_t R_t - \int_0^t 1_{\{L_s = R_s\}} \mathrm{d}s,$$
$$\text{(ii)} \quad |L_t - R_t| - 2\int_0^t 1_{\{L_s = R_s\}} \mathrm{d}s,$$

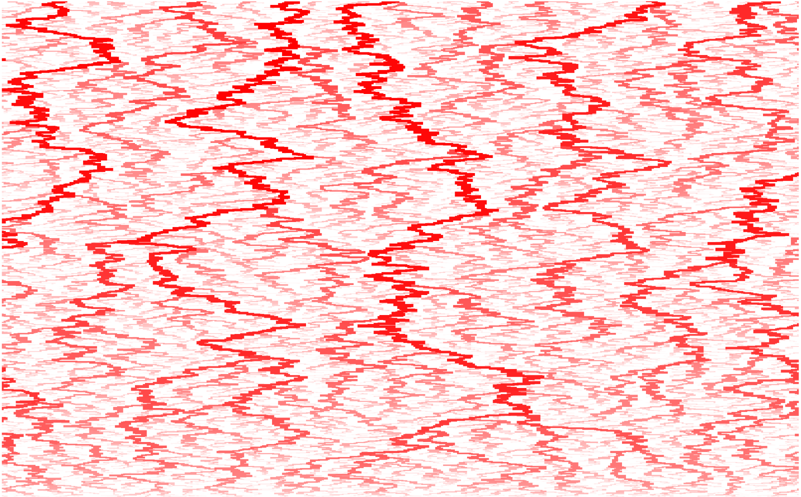
are martingales w.r.t. $(\mathcal{F}_t)_{t\geq 0}$.

The left-right Brownian web



This gives rise to a coupled left and right Brownian web $(\mathcal{W}^{l}, \mathcal{W}^{2})$.

The left-right Brownian web



This gives rise to a coupled left and right Brownian web $(\mathcal{W}^l, \mathcal{W}^2)$.

Hopping construction of the Brownian net

By definition, an *intersection time* of two paths π_1, π_2 is a time $t > \sigma_{\pi_1} \vee \sigma_{\pi_2}$ such that $\pi_1(t) = \pi_2(t)$.
We may concatenate two paths at an intersection time by putting

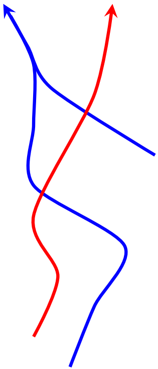
$$\pi(s) := \begin{cases} \pi_1(s) & \big(s \in [\sigma_{\pi_1}, t]\big), \\ \pi_2(s) & \big(s \in [t, \infty]\big). \end{cases}$$

Let $\mathcal{D} \subset \mathbb{R}^2$ be deterministic, countable, and dense.
Let $\mathcal{W}^{\mathrm{l}}(\mathcal{D})$ and $\mathcal{W}^{\mathrm{r}}(\mathcal{D})$ denote the sets of paths in $\mathcal{W}^{\mathrm{l}}$ and $\mathcal{W}^{\mathrm{r}}$ started from $\mathcal{D}$.

Let $\mathtt{Hop}\big(\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})\big)$ denote the smallest set containing $\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})$ that is closed under concatenation of open paths at intersection times.

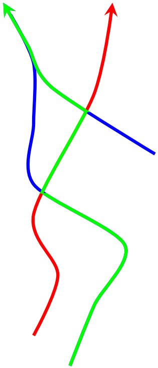
Hopping construction $\mathcal{N} := \overline{\mathtt{Hop}\big(\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})\big)}$.

Hopping construction of the Brownian net



Paths in $\mathcal{W}^{\mathrm{l}}(\mathcal{D}) \cup \mathcal{W}^{\mathrm{r}}(\mathcal{D})$.

Hopping construction of the Brownian net



A path in $\texttt{Hop}\big(\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})\big)$.

Hopping construction of the Brownian net

Let $(\mathcal{W}^{\rm l}(\mathcal{D}), \mathcal{W}^{\rm r}(\mathcal{D}))$ be sticky Brownian webs with drifts -1 and $+1$ and stickiness parameter 1.

Let $(\mathcal{W}^1, \mathcal{W}^2)$ be sticky Brownian webs with drift 0 and stickiness parameter 1.

It follows from results in **[Schertzer, Sun, & S. '14]** that

$$\mathcal{N} := \overline{\mathtt{Hop}\big(\mathcal{W}^{\rm l}(\mathcal{D}) \cup \mathcal{W}^{\rm r}(\mathcal{D})\big)} = \overline{\mathtt{Hop}\big(\mathcal{W}^1(\mathcal{D}) \cup \mathcal{W}^2(\mathcal{D})\big)}.$$

The proof that $\mathbb{P}\big[S^\sigma_\varepsilon(\mathcal{V}_\varepsilon)\in\cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[\mathcal{N}\in\cdot\,\big]$ consists of three steps:

I. Show tightness of the laws $\big\{\mathbb{P}\big[S^\sigma_\varepsilon(\mathcal{V}_\varepsilon)\in\cdot\,\big] : 0<\varepsilon\leq 1\big\}$.

By Shorohod's representation theorem, this allows us to select a subsequence such that

$$S^\sigma_\varepsilon(\mathcal{U}^i) \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{W}^i \quad (i=1,2) \quad \text{and} \quad S^\sigma_\varepsilon(\mathcal{V}_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{N}_*.$$

Setting $\mathcal{N} := \overline{\mathtt{Hop}\big(\mathcal{W}^1(\mathcal{D})\cup\mathcal{W}^2(\mathcal{D})\big)}$, it then remains to show that:

$$\text{II. } \mathcal{N}\subset\mathcal{N}_* \quad \text{and} \quad \text{III. } \mathcal{N}_*\subset\mathcal{N}.$$

The lower bound II. follows from the hopping construction.

The upper bound

Once we have the lower bound $\mathcal{N} \subset \mathcal{N}_*$, in order to prove the upper bound $\mathcal{N}_* \subset \mathcal{N}$, it should suffice to prove $\mathbb{E}[\mathcal{N}_*] \leq \mathbb{E}[\mathcal{N}]$. The process $(\xi_t)_{t\geq 0}$ defined as

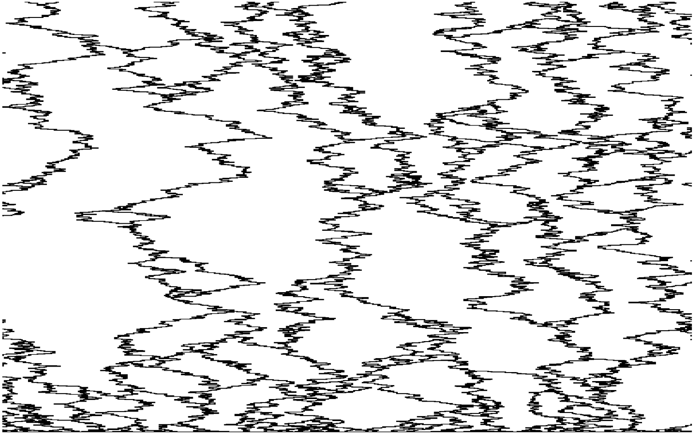
$$\xi_t := \{\pi(t) : \pi \in \mathcal{N},\ \sigma_\pi = 0\} \qquad (t \geq 0)$$

is called the *branching-coalescing point set*.
It is known that $(\xi_t)_{t\geq 0}$ *comes down from infinity:*

$$\mathbb{E}\big[|\xi_t \cap [a,b]|\big] < \infty \quad \forall t > 0,\ a < b,$$

and has a reversible invariant law that is the law of a Poisson point set with intensity 2.

The branching-coalescing point set



The branching-coalescing point set.

The upper bound

Let ξ_0 be a Poisson point set with intensity 2, independent of $\mathcal{N}$ and $\mathcal{N}_*$, and set

$$\begin{aligned} \xi_t &:= \big\{\pi(t) : \pi \in \mathcal{N},\ \sigma_\pi = 0,\ \pi(0) \in \xi_0\big\}, \\ \xi^*_t &:= \big\{\pi(t) : \pi \in \mathcal{N}_*,\ \sigma_\pi = 0,\ \pi(0) \in \xi_0\big\} \end{aligned} \qquad (t \geq 0)$$

Then ξ_t is a Poisson point set with intensity 2.
By a finite energy argument, to prove the upper bound $\mathcal{N}_* \subset \mathcal{N}$, it suffices to prove

$$\mathbb{E}\big[|\xi^*_t \cap [a,b]|\big] \leq 2(b-a) \qquad (t > 0,\ a < b).$$

The upper bound

In the discrete setting, for given $X_0 \subset \mathbb{Z}$, we define a system of *branching and coalescing random walks* by

$$X_t^\varepsilon := \{\pi(t) : \pi \in \mathcal{V}_\varepsilon,\ \sigma_\pi = 0,\ \pi(0) \in X_0\}.$$

One may hope that this has a product measure as reversible invariant law, but this is not true.

It becomes true, however, if we slightly change the definition of $\mathcal{V}_\varepsilon$.

The upper bound

Independently for all x, y, t in $\mathbb{Z}$, draw an arrow from (x, t) to $(y, t+1)$ with probability

$$1 - e^{-ra(y-x)} \quad \text{where} \quad 1 - \varepsilon =: \frac{r}{e^r - 1},$$

and then condition on the event that each $(x, t) \in \mathbb{Z}^2$ is the starting point of at least one arrow. Let $\tilde{\mathcal{V}}_\varepsilon$ denote the resulting discrete net and let $(\tilde{X}^\varepsilon_t)_{t \geq 0}$ be the associated branching and coalescing random walks. Then:

- $\tilde{\mathcal{V}}_\varepsilon$ stochastically dominates $\mathcal{V}_\varepsilon$.
- $(\tilde{X}^\varepsilon_t)_{t \geq 0}$ has product measure with intensity $1 - e^{-r} = 2\varepsilon + O(\varepsilon^2)$ as reversible invariant law.

Tightness

Our proofs of II. the lower bound and III. the upper bound are dependent on I. tightness.

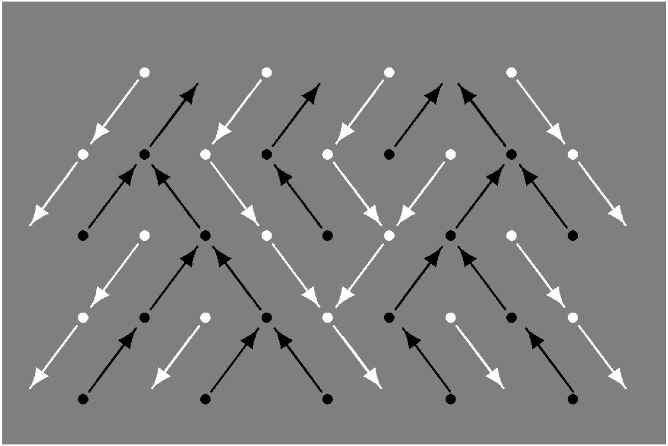
This is not just a formality. Our proofs of II. and III. have not used the condition $\sum_{k\in\mathbb{Z}} a(k)|k|^{3+\delta} < \infty$.

Yet, we know that as soon as $\limsup_{n\to\infty} A(n)n^3 > 0$, with $A(n) := \sum_{k=m}^{\infty} a(k)$, there are macroscopic jumps in the limit.

It is hard to imagine what else could go wrong,
so we believe $\lim_{n\to\infty} A(n)n^3 = 0$ should be sufficient.

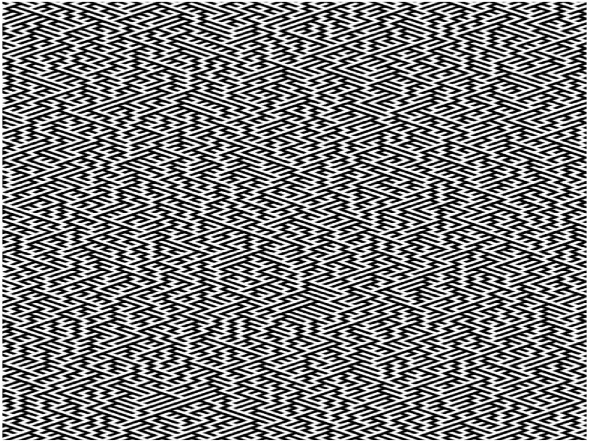
But that is not a proof.

The dual picture



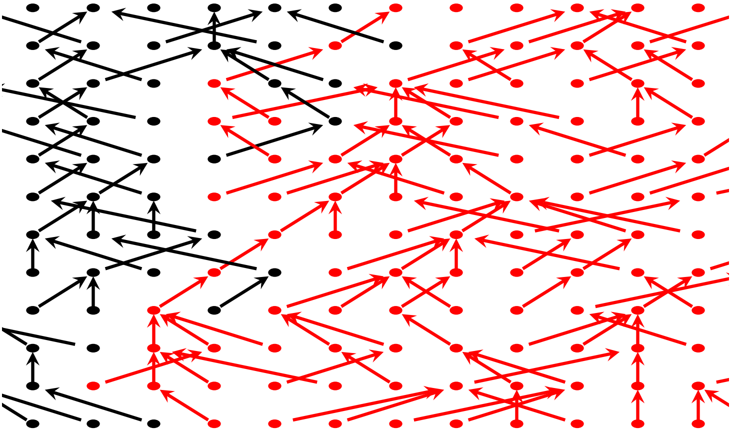
Associated to each discrete nearest-neighbour web $\mathcal{U}$ there is a *dual* discrete nearest-neighbour web $\hat{\mathcal{U}}$.

The dual picture



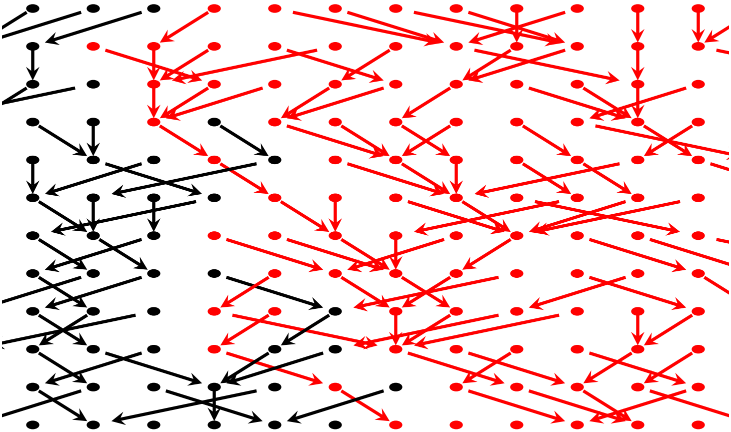
Likewise, associated with each Brownian web $\mathcal{W}$ there is a dual web $\hat{\mathcal{W}}$.

The dual picture



In the discrete non-nearest neighbour case, we can make the dual web visible by looking at the set of all (x, t) with $t \leq 0$ such that the path π that starts at (x, t) satisfied $\pi(0) \geq 0$.

The dual picture



If we turn the picture upside down, then this defines a Markov chain: a *discrete time voter model* (or *spatial Moran model*).

The discrete time voter model

Let $\{0,1\}^{\mathbb{Z}}$ be the space of functions $y : \mathbb{Z} \to \{0,1\}$.
Let $\big(\omega(x,t)\big)_{(x,t)\in\mathbb{Z}^2}$ be i.i.d. with law a.
The *discrete time voter model* (or *spatial Moran model*) with initial state $Y_0 \in \{0,1\}^{\mathbb{Z}}$ is defined as

$$Y_t(x) := Y_{t-1}\big(x - \omega(x,t)\big) \qquad (t \geq 1).$$

We are especially interested in the initial state

$$Y_0(k) := 1 \quad (k < 0) \quad \text{and} \quad Y_0(k) := 0 \quad (k \geq 0).$$

Define $L_t, R_t \in \mathbb{Z} + \frac{1}{2}$ by

$$L_t := \sup\big\{z \in \mathbb{Z} + \tfrac{1}{2} : Y_t(k) = 1 \ \forall k < z\big\},$$
$$R_t := \inf\big\{z \in \mathbb{Z} + \tfrac{1}{2} : Y_t(k) = 0 \ \forall z < k\big\}$$

If rescaled discrete webs converge to the Brownian web, then

$$\mathbb{P}\big[\big(\varepsilon L_{\sigma^2\varepsilon^{-2}t}, \varepsilon R_{\sigma^2\varepsilon^{-2}t}\big)_{t\geq 0} \in \cdot\,\big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[(B_t, B_t)_{t\geq 0} \in \cdot\,\big],$$

for some standard Brownian motion $(B_t)_{t\geq 0}$.

The discrete time biased voter model

Let $\big(\omega^1(x,t)\big)_{(x,t)\in\mathbb{Z}^2}$ be i.i.d. with law a and let $\big(\omega^2(x,t)\big)_{(x,t)\in\mathbb{Z}^2}$ be obtained by resampling an ε-fraction of the points $(x,t)\in\mathbb{Z}^2$. The *biased* discrete time voter model is defined as

$$Y_t(x) := Y_{t-1}\big(x-\omega^1(x,t)\big) \vee Y_{t-1}\big(x-\omega^2(x,t)\big) \qquad (t\geq 1).$$

If rescaled discrete *nets* converge to the Brownian web, then

$$\mathbb{P}\Big[\big(\varepsilon L_{\sigma^2\varepsilon^{-2}t}, \varepsilon R_{\sigma^2\varepsilon^{-2}t}\big)_{t\geq 0} \in \cdot\,\Big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\big[(B_t+t, B_t+t)_{t\geq 0} \in \cdot\,\big] \quad (1)$$

for some standard Brownian motion $(B_t)_{t\geq 0}$.

Conversely, to prove tightness of the law of diffusively rescaled discrete nets, it suffices to prove (1).

Note that (1) fails if there are macroscopic jumps in the limit!

Interface tightness

Let $S^{10} := \{y \in \{0,1\}^{\mathbb{Z}} : \lim_{k\to-\infty} y(k) = 1,\ \lim_{k\to\infty} y(k) = 0\}$.
For $y, y' \in S^{10}$, write $y \sim y'$ if y' is a translation of y and let $\overline{y} := \{y' \in S^{01} : y \sim y'\}$ denote the equivalence class containing y.

By definition, *interface tightness* holds if $(\overline{Y}_t)_{t\geq 0}$ is a positive recurrent Markov chain with state space $\overline{S}^{10} := \{\overline{y} : y \in S^{10}\}$.

[Cox & Durrett '95] proved interface tightness for continuous time voter models with $\sum_{k\in\mathbb{Z}} a(k)|k|^3 < \infty$.

[Belhaouari, Mountford, & Valle '07] relaxed this to $\sum_{k\in\mathbb{Z}} a(k)k^2 < \infty$, which is optimal.

[Sun, S., & Yu '19] proved interface tightness for biased voter models.

The weighted midpoint

For each $y \in S^{10}$, let $M(y) \in \mathbb{Z} + \frac{1}{2}$ be defined by

$$\#\big\{k < M(y) : y(k) = 0\big\} = \#\big\{k > M(y) : y(k) = 1\big\}.$$

[Sun, S., & Yu '21] showed that for biased voter models,

$$\mathbb{P}\Big[\big(\varepsilon M(Y_{\sigma^2\varepsilon^{-2}t})\big)_{t\geq 0} \in \cdot\Big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\Big[(B_t + t)_{t\geq 0} \in \cdot\Big],$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

Moreover, the measure-valued process $\mu^\varepsilon_t := \varepsilon \sum_{k\in\mathbb{Z}} Y_{\sigma^2\varepsilon^{-2}t}\delta_{\varepsilon k}$

converges weakly in law as a process to

$$\mu_t(\mathrm{d}x) := 1_{(-\infty,B_t+t]}(x)\mathrm{d}x.$$

All this needs only $\sigma^2 := \sum_k a(k)k^2 < \infty$.

The weighted midpoint

Now we "only" needed to show that if $\sum_{k\in\mathbb{Z}} a(k)|k|^{3+\delta} < \infty$, then the previous statement can be boosted to

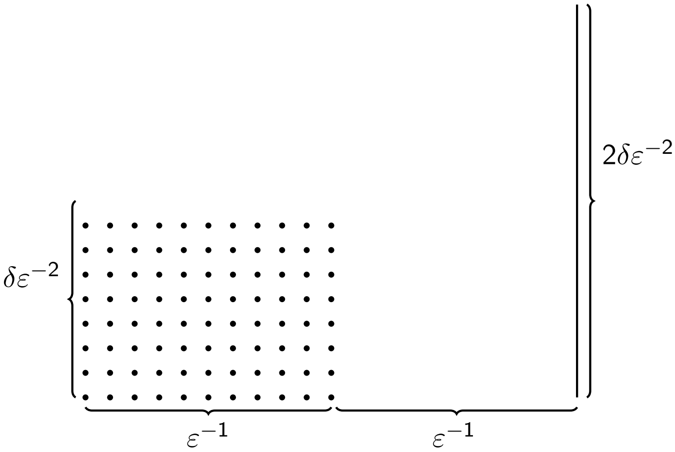
$$\mathbb{P}\Big[\big(\varepsilon L_{\sigma^2\varepsilon^{-2}t}, \varepsilon R_{\sigma^2\varepsilon^{-2}t}\big)_{t\geq 0} \in \cdot\,\Big] \underset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}\Big[(B_t + t, B_t + t)_{t\geq 0} \in \cdot\,\Big].$$

We could do this based on a reasonable conjecture for the tail of the equilibrium distribution of $R_t - L_t$.

Unfortunately, we did not manage to solve this conjecture.

Back to the drawing board.

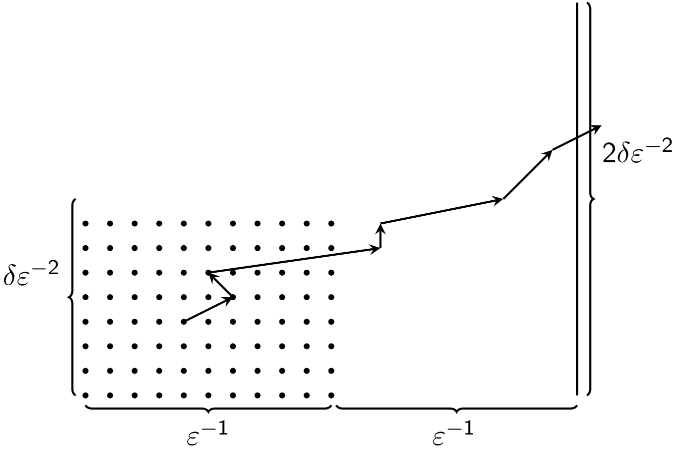
A multiscale argument



[Belhaouari, Mountford, Sun, & Valle '06]

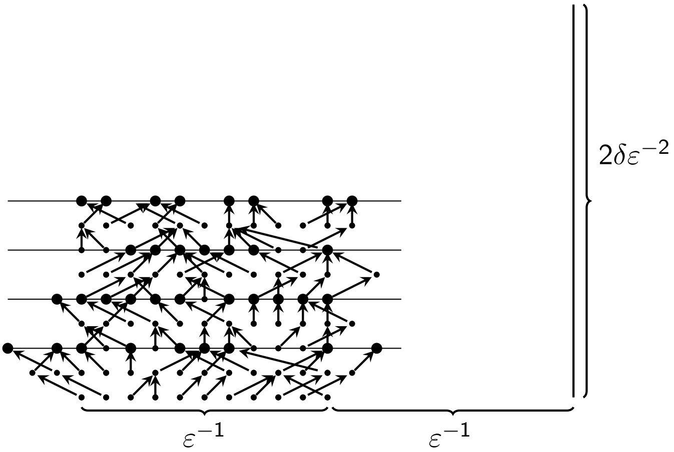
used a multiscale argument for discrete webs.

A multiscale argument



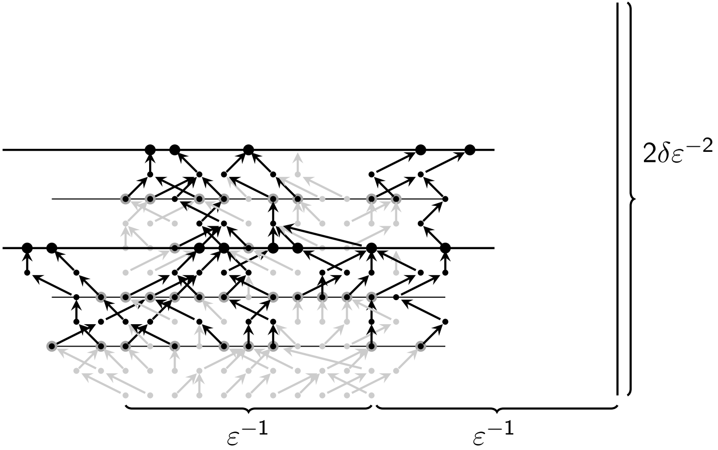
We need to show that the probability of this event tends to zero for $\delta \to 0$, uniformly in ε.

A multiscale argument



In each step, we let the coalescing random walks of two subsequent times evolve until the next time.

A multiscale argument



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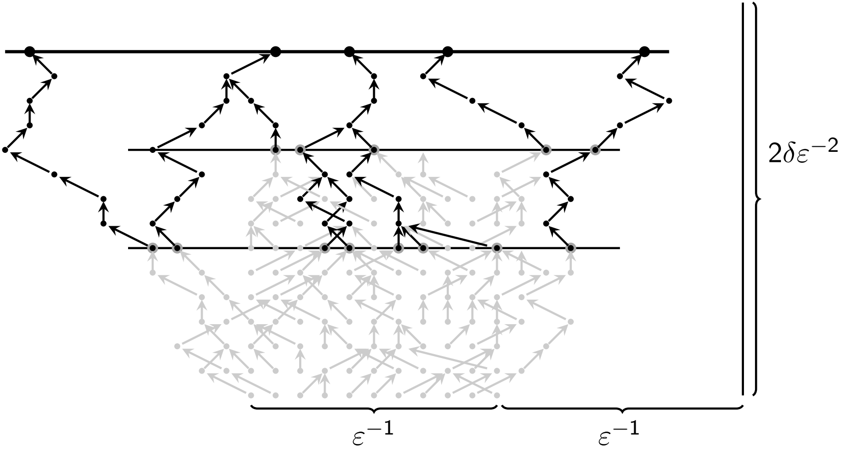
A multiscale argument



In each step, we let the coalescing random walks of two subsequent times evolve until the next time.

A multiscale argument

By controlling the number of particles and their displacement in each step, Belhaouari, Mountford, Sun, & Valle in 2006 were able to control the maximal displacement of all paths started in the block of size $\varepsilon^{-1} \times \delta\varepsilon^{-2}$.

The argument is a bit lossy, which is why they needed the condition

$$\sum_{k \in \mathbb{Z}} a(k)|k|^{3+\delta} < \infty \quad \text{for some } \delta > 0,$$

that is a bit stronger than the optimal condition

$$\limsup_{n \to \infty} A(n)n^3 = 0 \quad \text{with} \quad A(n) := \sum_{k=m}^{\infty} a(k).$$

A multiscale argument

In the presence of branching, the argument (that is already quite heavy) becomes a lot harder, which is why we could not pull it through for many years.

The way we managed, in the end, is quite interesting, but does not fit in the time frame of this talk.