A numerical search for intertwining relations

Jan M. Swart

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Work in progress!

Jan M. Swart A numerical search for intertwining relations

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A relation between square matrices of the form

$$PK = KQ \tag{(*)}$$

is called an *intertwining relation* and K is the *intertwiner*. If K is invertible, then we can rewrite (\star) as

$$Q = K^{-1}PK$$
 or $P = KQK^{-1}$.

This says that P and Q are *similar*.

By definition, P is diagonalisable if K can be chosen such that Q is diagonal.

An
$$d \times d$$
 matrix P is a probability kernel if
(i) $P(x, y) \ge 0$ $(1 \le x, y \le d)$,
(ii) $\sum_{y=1}^{d} P(x, y) = 1$ $(1 \le x \le d)$.

Its *n*-th power P^n describes the *n*-step transition probabilities of the Markov chain with transition kernel P.

If we can diagonalise P, then we have good control over its powers, since

$$P^n = KQ^n K^{-1} \qquad (t \ge 0),$$

and it is trivial to calculate the *n*-th power of a diagonal matrix.

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In practice, it can be hard to have good control over the eigenvalues of P and the intertwiner K that diagonalises P.

Also, by diagonalising P, we leave the space of probability kernels, so in a sense we forget about the special property that P is a probability kernel (in particular, the nonnegativity of its elements).

In view of this, as an alternative to diagonalisation, we can look for intertwining relations of the form

(i)
$$PK = KQ$$
 or (ii) $KP = QK$,

where P, Q, K are all probability kernels, and Q is "as simple as possible".

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Note that

(i)
$$PK = KQ$$
 implies $P^nK = KQ^n$
(ii) $KP = QK$ implies $KP^n = Q^nK$
($n \ge 0$).

In case (i), let's say that Q is intertwined on top of P and in case (ii), let's say that Q is intertwined below P.

PK = KQ implies $K^{-1}P = QK^{-1}$, but K^{-1} is in general not a probability kernel.

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Continuous time

A Markov semigroup is a family $(P_t)_{t\geq 0}$ of square probability kernels such that $t \mapsto P_t$ is continuous, $P_0 = 1$, and $P_s P_t = P_{s+t}$ $(s, t \geq 0)$.

Each Markov semigroup is of the form

$$P_t = e^{tG} := \sum_{k=0}^{\infty} \frac{1}{k!} t^k G^k,$$

where the generator G satisfies

(i)
$$G(x, y) \ge 0 \quad \forall x \ne y$$
,
(ii) $\sum_{y=1}^{n} G(x, y) = 0 \quad \forall x$.
For $x \ne y$, we call $G(x, y)$ the *rate* of jumps from x to y.

For semigroups $(P_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ with generators G, H, one has

(i) GK = KH implies $P_tK = KQ_t$ (ii) KG = HK implies $KP_t = Q_tK$ $(t \ge 0).$

In case (i), we say that H is intertwined on top of G and in case (ii), we say that H is intertwined below G.

Image: A image: A

Assume that Q is intertwined on top of P, i.e., PK = KQ.

Then it is possible to construct a Markov chain $(X_n, Y_n)_{n \ge 0}$ such that

$$\mathbb{P}[Y_0 \in \cdot | X_0] = K(X_0, \cdot)$$
 a.s.

implies that

$$\mathbb{P}\big[Y_n \in \,\cdot\, \big|\, (X_k)_{0 \leq k \leq n}\big] = \mathcal{K}(X_n, \,\cdot\,) \quad \text{a.s.} \quad (n \geq 0).$$

Moreover (note that Y is *autonomous* but X is not):

$$\begin{split} \mathbb{P}\big[X_{n+1} = x \big| \, (X_k)_{0 \leq k \leq n}\big] &= P(X_n, x) \quad \text{a.s.}, \\ \mathbb{P}\big[Y_{n+1} = y \big| \, (X_k, Y_k)_{0 \leq k \leq n}\big] &= Q(Y_n, y) \quad \text{a.s.} \end{split}$$

An analogue result holds on the continuous-time case. [Rogers & Pitman '81, Fill '92]

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Consider a continuous-time process on $\{0, \ldots, d\}$ that jumps with rate β_k from k - 1 to k and with rate δ_k from k to k - 1 $(1 \le k \le d)$.

Assume that $\delta_d = 0$ so that d is a trap.



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In **[Diaconis & Miclos '09]** It has been shown that it is possible to intertwine the generator H of a pure birth process below G, whose jump rates $\lambda_d > \cdots > \lambda_1$ are the negatives of the nontrivial eigenvalues of G.



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The proof is based on a repeated application of the Perron-Frobenius theorem.



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In '10, I showed that it is similarly possible to intertwine the generator H of a pure birth process *on top of* the generator G of a birth-and-death process.



The example of birth-and-death chains shows that:

- For some transition kernels P, it is possible to find a simpler transition kernel Q and an intertwining kernel K such that PK = KQ (Q on top of P) or KP = QK (Q below P).
- ▶ Using the simpler kernel *Q*, it is possible to get information about the long-time behaviour of the Markov chain with transition kernel *P* (such as the time till absorption).
- Even though Q is not diagonal, there is a relation between Q and the eigenvalues of P.

(In our example, $P = P_t$ and $Q = Q_t$ are the transition kernels of a continuous-time Markov process.)

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Questions remain:

- Is there a similar picture for discrete-time birth-and-death chains?
- To what extent does this generalise beyond birth-and-death chains?
- For example, if P has one absorbing state, then can one always choose K and Q so that they are triangular?
- ▶ What if *P* does not have an absorbing state but is ergodic?

To investigate these and related questions, we will look at numerical methods that aim to find Q and K given P.

We will focus on the problem of finding Q that are intertwined *on top* of P.

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A simple idea:

Let *P* be a probability kernel of size $d \times d$. Let 1 denote the identity matrix of size $d \times d$. Let K_0, K_1, \ldots be inductively defined by

$$egin{array}{ccc} \mathcal{K}_0:=1 & ext{and} & \mathcal{K}_{s+1}:=\mathcal{K}_s+\mathcal{P}\mathcal{K}_s-\mathcal{K}_s\mathcal{Q}_s & (s\geq 0), \end{array}$$

where $Q_s = Q(P, K_s)$ is some function of P and K_s . Assume that

$$K_s \xrightarrow[s \to \infty]{} K$$
 and $Q_s \xrightarrow[s \to \infty]{} Q$.

Then $PK - KQ = \lim_{s \to \infty} (PK_s - K_sQ_s) = \lim_{s \to \infty} (K_{s+1} - K_s) = 0.$

How to choose the function $Q(P, K_s)$?

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Here's an idea:

Let \mathcal{K} be the space of probability kernels of size $d \times d$. Let $[d] := \{1, \ldots, d\}$. Fix $Z \subset \{(x, y) \in [d]^2 : x \neq y\}$. Set

$$\mathcal{K}_{\mathbf{Z}} := \big\{ K \in \mathcal{K} : K(x, y) = 0 \ \forall (x, y) \in \mathbf{Z} \big\}, \\ \mathcal{C}_{\mathbf{Z}}(P, K) := \big\{ Q \in \mathcal{K} : K' := K + PK - KQ \in \mathcal{K}_{\mathbf{Z}} \big\},$$

and define

$$\mathcal{Q}_{Z}(P,K) :=$$
 the unique minimiser of
 $Q \mapsto \sum_{x \neq y} Q(x,y)$ on $\mathcal{C}_{Z}(P,K)$.

Assuming the minimiser exists and is unique!

An evolution equation

Recall $K_{s+1} := K_s + PK_s - K_sQ_s$.

- ► Need to choose Q_s such that K_{s+1} := K_s + PK_s K_sQ_s is a probability kernel.
- ▶ By minimising $Q \mapsto \sum_{x \neq y} Q(x, y)$ we try to choose Q_s as "simple" as possible.
- Without further restrictions on K_{s+1} , the minimiser is $Q_s = 1$, which gives the trivial evolution $K_{s+1} := PK_s$.
- By requiring that K_{s+1}(x, y) = 0 for (x, y) ∈ Z, we can use our intuition about what the intertwiner should look like.

This is (so far) *nonrigorous:* no proof that the minimiser exists, or is unique, or that the limits $K := \lim_{s\to\infty} K_s$ and $Q := \lim_{s\to\infty} Q_s$ exist.

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An evolution equation

Given probability kernels $P \in \mathcal{K}$ and $K \in \mathcal{K}_Z$, $\mathcal{C}_Z(P, K)$ is the space of all $d \times d$ matrices such that:

(i)
$$Q(x,y) \ge 0$$
 $\forall x, y,$
(ii) $\sum_{y=1}^{d} Q(x,y) = 1$ $\forall x,$
(iii) $KQ(x,y) = K(x,y) + PK(x,y)$ $\forall (x,y) \in Z$
(iv) $KQ(x,y) \le K(x,y) + PK(x,y)$ $\forall (x,y) \notin Z$

Here (i) and (ii) say that Q is a probability kernel, while (iii) and (iv) say that K' := K + PK - KQ is nonnegative with K'(x, y) = 0 for $(x, y) \in Z$.

The fact that $\sum_{y} K'(x, y) = 1 \ \forall x$ follows from the fact that P, K, and Q have this property.

An evolution equation

To calculate $Q = Q_Z(P, K)$, we have to minimise

$$Q\mapsto \sum_{x
eq y}Q(x,y)$$

subject to the constraints

(i)
$$Q(x,y) \ge 0$$
 $\forall x, y,$
(ii) $\sum_{y=1}^{d} Q(x,y) = 1$ $\forall x,$
(iii) $KQ(x,y) = K(x,y) + PK(x,y)$ $\forall (x,y) \in \mathbb{Z},$
(iv) $KQ(x,y) \le K(x,y) + PK(x,y)$ $\forall (x,y) \notin \mathbb{Z}.$

This is a standard exercise in linear optimisation. There exist fast algorithms that give you a solution, if it exists.

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I have written a couple of scripts in the scientific programming language GNU Octave that numerically solve the equation

$$K_0:=1$$
 and $K_{s+1}:=K_s+PK_s-K_sQ_s$ $(s\geq 0)$ $(\star).$

with $Q_s = Q_{\mathbb{Z}}(P, K_s)$.

The input are a probability kernel P of size $d \times d$ and a matrix Z of size $d \times d$ containing only zeros and ones, where Z(x, y) = 1 means that $(x, y) \in Z$.

The program runs (*) until $PK_s - K_sQ_s$ is close enough to zero.

These scripts are available from my homepage with instructions on how to use them, so you can give them a try if you wish.

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Let P be the transition kernel of a *discrete*-time Markov chain on $\{0, \ldots, d\}$ that jumps with probability b_k from k - 1 to k, with probability c_k from k to k and with probability d_k from k to k - 1.

Assume that $c_d = 1$ so that d is a trap.



We are looking for an intertwining relation of the form

PK = KQ

where K(k, k + 1) = 0 for k = 0, ..., d - 1and Q is as simple as possible.



Numerically, we find that such an intertwining really exists. Here $1 = \gamma_0 \ge \cdots \ge \gamma_d$ are the eigenvalues of P.



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Numerically, we find that such an intertwining really exists. Here $1 = \gamma_0 \ge \cdots \ge \gamma_d$ are the eigenvalues of P. I am cheating a bit: this works if P is a *lazy kernel* which means that $P = \frac{1}{2}(1 + P')$ for some kernel P', which guarantees that $\gamma_i \ge 0 \ \forall i = 0, \dots, d$.



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0.5000 0.2500 0 0	0.5000 0.5000 0.2500 0 0	0 0.2500 0.5000 0.2500 0	0 0.2500 0.5000 0	0 0 0.2500 1.0000				
octave:24> Z =	Z							
0 1 0 0 0 0 0 0 0 0	0 0 1 0 0 1 0 0 0 0	0 0 0 1 0						
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octave:25> step Q =						
0.5000 0.5000 0 0.7500 0 0 0 0 0 0	0 0.2500 0.7500 0. 0 0. 0	0 0 2500 0 7500 0.250 0 1.000	0 0 0			
К =						
1.0000 0 0.2500 0.7500 0 0.2500 0 0 0 0	0 0.7500 0.2500 0. 0	0 0 0 0 7500 0 0 1.000	0 0 0 0			
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octave:26> Q =	step							
0.6250 0 0 0	0.3750 0.7500 0 0 0	0 0.2500 0.7500 0 0	0 0.2500 0.6667 0	0 0 0.3333 1.0000				
К =								
1.0000 0.4687 0.0625 0 0	0 0.5312 0.3750 0.0625 0	0 0.5625 0.3750 0	0 0 0.5625 0	0 0 0 1.0000				
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octave:27> Q =	step							
0.7344 0 0 0	0.2656 0.7353 0 0	0 0.2647 0.7500 0	0 0.2500 0.5556	0 0 0.4444				
0 K =	0	0	0	1.0000				
1.0000 0.6245 0.1650 0.0156 0	0 0.3755 0.4186 0.1415 0	0 0 0.4164 0.4053 0	0 0 0.4375 0	0 0 0 1.0000				
octave:28>								0
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octave:28> Q =	step							
0.8123	0.1877	0	0	0				
0	0.7228	0.2772	0	0				
0	0	0.7373	0.2627	0				
0	0	0	0.4286	0.5714				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.7208	0.2792	0	0	0				
0.2735	0.4236	0.3028	0	0				
0.0520	0.2117	0.3740	0.3623	0				
0	0	0	0	1.0000				
octave:29>	I							0
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octave:29	> step							
0.8604	0.1396	0	0	0				
0	0.7289	0.2711	0	0				
0	0	0.7009	0.2991	0				
0	0	0	0.3099	0.6901				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.7794	0.2206	0	0	0				
0.3682	0.4112	0.2206	0	0				
0.1016	0.2619	0.3171	0.3193	0				
0	0	0	0	1.0000				
octave:30	>							0
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octave:30> Q =	step							
0.8897 0 0 0	0.1103 0.7500 0 0 0	0 0.2500 0.6382 0 0	0 0.3618 0.2170 0	0 0 0.7830 1.0000				
К =								
1.0000 0.8177 0.4449 0.1541 0	0 0.1823 0.3884 0.2880 0	0 0.1666 0.2630 0	0 0 0.2949 0	0 0 0 1.0000				
octave:31>	•							0
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octave:31> Q =	step							
0.9089	0.0911 0.7715 0	0 0.2285 0.5575 0	0 0.4425 0.1523	0 0 0.8477				
К =	0	0	0	1.0000				
1.0000 0.8446 0.5060 0.2023 0 octave:32>	0 0.1554 0.3600 0.2929 0	0 0.1340 0.2237 0	0 0 0.2811 0	0 0 0 1.0000				0
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octave:32> Q =	step							
0.9223	0.0777	0	0	0				
0	0.7844	0.2156	0	0				
0	0	0.4757	0.5243	0				
0	0	0	0.1106	0.8894				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.8644	0.1356	0	0	0				
0.5540	0.3304	0.1156	0	0				
0.2434	0.2839	0.1995	0.2732	0				
0	0	0	0	1.0000				
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octave:33> Q =	step							
0.9322	0.0678	0	0	0				
0	0.7869	0.2131	0	0				
0	0	0.4090	0.5910	0				
0	0	0	0.0851	0.9149				
0	0	0	0	1.0000				
K =								
1.0000	0	0	0	0				
0.8793	0.1207	0	0	0				
0.5915	0.3029	0.1056	0	0				
0.2767	0.2685	0.1860	0.2687	0				
0	0	0	0	1.0000				
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octave:34> Q =	⊳ step							
0.9397	0.0603	0	0	0				
0	0.7813	0.2187	0	0				
0	0	0.3636	0.6364	0				
0	0	0	0.0696	0.9304				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.8906	0.1094	0	0	0				
0.6205	0.2793	0.1002	0	0				
0.3029	0.2520	0.1791	0.2660	0				
0	0	0	0	1.0000				
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octave:35> Q =	step							
0.9453	0.0547	0	0	0				
0	0.7710	0.2290	0	0				
0	0	0.3366	0.6634	0				
0	0	0	0.0600	0.9400				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.8991	0.1009	0	0	0				
0.6425	0.2601	0.0974	0	0				
0.3231	0.2370	0.1757	0.2642	0				
0	0	0	0	1.0000				
octave:36>	I							0
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octave:36> g Q =	step							
0.9496	0.0504	0	0	0				
0	0./580	0.2414	0 6701	0				
	0	0.3219	0.0781	0 0/63				
0	0	0	0.0557	1.0000				
К =								
1.0000	0	0	0	0				
0.9055	0.0945	0	0	0				
0.6592	0.2449	0.0959	0	0				
0.3385	0.2244	0.1741	0.2630	0				
0	0	0	0	1.0000				
octave:37>								0
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octave:37> Q =	step							
0.9528 0 0	0.0472 0.7462 0	0 0.2538 0.3144	0 0 0.6856	0 0 0				
0 0	0 0	0	0.0493 0	0.9507 1.0000				
К =								
1.0000 0.9104 0.6718 0.3501 0	0 0.0896 0.2332 0.2144 0	0 0.0951 0.1734 0	0 0 0.2621 0	0 0 0 1.0000				
octave:38>	I .							0
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octave:38> Q =	step							
0.9552	0.0448	0	0	0				
0	0.7349	0.2651	0	0				
0	0	0.3107	0.6893	0				
0	0	0	0.0462	0.9538				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9139	0.0861	0	0	0				
0.6811	0.2243	0.0946	0	0				
0.3587	0.2066	0.1732	0.2615	0				
0	0	0	0	1.0000				
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octave:39> Q =	step							
0.9570	0.0430	0	0	0				
0	0.7252	0.2748	0	0				
0	0	0.3089	0.6911	0				
0	0	0	0.0440	0.9560				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9166	0.0834	0	0	0				
0.6880	0.2176	0.0943	0	0				
0.3650	0.2007	0.1731	0.2611	0				
0	0	0	0	1.0000				
octave:40>	I							0
twine : bash	× kern :	bash \times ke	ern : octave	-gui ×				

$\overline{}$			kern : oo	tave-gui –	- Konsole			~ ^ ×
File Edit	View Bo	okmarks	Plugins	Settings	Help			
New Tal	b 👖 Spli	t View 🗸				🗅 Сору	Paste	Q Find
octave:40> Q =	step							
0.9583	0.0417	0	0	0				
0	0.7173	0.2827	0	0				
0	0	0.3082	0.6918	0				
0	0	0	0.0424	0.9576				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9185	0.0815	0	0	0				
0.6931	0.2127	0.0942	0	0				
0.3697	0.1963	0.1732	0.2608	0				
0	0	0	0	1.0000				
octave:41>	•							0
twine : bash	× kern :	bash $ imes$ ke	ern : octave	-aui ×				
				3				

$\overline{}$			kern : oo	tave-gui –	- Konsole			~ ^ ×
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octave:41> Q =	step							
0.9593	0.0407	0	0	0				
0	0.7110	0.2890	0	0				
0	0	0.3080	0.6920	0				
0	0	0	0.0412	0.9588				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9200	0.0800	0	0	0				
0.6969	0.2090	0.0941	0	0				
0.3732	0.1930	0.1733	0.2605	0				
0	0	0	0	1.0000				
octave:42>	•							0
twine : bash	× kern :	bash $ imes$ ke	ern : octave	-qui ×				
				5				

$\overline{}$			kern : oo	tave-gui –	- Konsole			~ ^ ×
File Edit	View B	ookmarks	Plugins	Settings	Help			
New Ta	ab 👖 Spl	it View 🗸				🗅 Сору	Paste	Q Find
octave:42> Q =	• step							
0.9600	0.0400	0	0	0				
0	0.7061	0.2939	0	0				
0	0	0.3079	0.6921	0				
0	0	0	0.0404	0.9596				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9210	0.0790	0	0	0				
0.6996	0.2063	0.0941	0	0				
0.3758	0.1906	0.1733	0.2604	0				
0	0	0	0	1.0000				
octave:43>	·							0
twine : bash	kern :	bash × ke	ern : octave	-gui ×				

				kern : o	ctave-gui –	– Konsole			~ ^ × `
File E	dit Vie	w Bo	okmarks	Plugins	Settings	Help			
📑 Ne	w Tab	Spli	t View 🗸				🗅 Сору	🗐 Paste	Q Find
octave: Q =	43> ste	p							
0.96	05 0.	0395	0	0	0				
	0 0.	7023	0.2977	0	0				
	0	0	0.3080	0.6920	0				
	0	0	0	0.0398	0.9602				
	0	0	0	0	1.0000				
К =									
1.00	00	0	0	0	0				
0.92	18 0.	0782	0	0	0				
0.70	16 0.	2043	0.0940	0	0				
0.37	76 0.	1887	0.1734	0.2602	0				
	0	0	0	0	1.0000				
octave:	44>								0
twine : b	ash ×	kern : l	ash imes k	ern : octave	e-gui ×				

$\overline{}$			kern : oo	tave-gui –	- Konsole			~ ^ ×
File Edit	View Bo	ookmarks	Plugins	Settings	Help			
📑 New Ta	b 🚺 Spli	it View 🗸				🗅 Сору	Paste	Q Find
octave:44> Q =	step							
0.9609	0.0391	0	0	0				
Θ	0.6995	0.3005	0	0				
0	0	0.3081	0.6919	0				
0	0	0	0.0393	0.9607				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9223	0.0777	0	0	0				
0.7031	0.2029	0.0940	0	0				
0.3790	0.1874	0.1735	0.2601	0				
0	0	0	0	1.0000				
octave:45>	•							0
twine : bash	× kern :	bash × ke	ern : octave	-aui ×				
				5				

Σ			kern : oo	tave-gui –	- Konsole			~ ^ ×
File Edit	View Bo	ookmarks	Plugins	Settings	Help			
📑 New Tal	b 👖 Spli	t View 🗸				🗅 Сору	Paste	Q Find
octave:45> Q =	step							
0.9612 0 0	0.0388 0.6973 0 0	0 0.3027 0.3082 0	0 0 0.6918 0.0390	0 0 0.9610 1 0000				
К =								
1.0000 0.9228 0.7042 0.3800 0	0 0.0772 0.2018 0.1864 0	0 0 0.0940 0.1735 0	0 0 0.2601 0	0 0 0 1.0000				
octave:46>	•							0
twine : bash	× kern :	bash × ke	ern : octave	-gui ×				

File Edit View Bookmarks Plugins Settings Help	Σ			kern : o	ctave-gui –	- Konsole			~ ^ × `
New Tab Image: Split View Paste Q Find Q = 0.9614 0.0386 0 0 0 0 .9614 0.0386 0 0 0 0 0 .9614 0.0386 0 0 0 0 0 .9614 0.0386 0 0 0 0 0 .00383 0.6917 0 0 0.9613 0 .0 0 0 0.10000 0 0 K = 1.0000 0 0 0 0 0.2311 0.0769 0 0 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 0 1.0000 0 octave: 47> Image: Note the set of the set	File Edit	View Bo	okmarks	Plugins	Settings	Help			
<pre>octave:46> step Q =</pre>	📑 New Tal	b 🔲 Spli	t View 🗸				🗅 Сору	Paste	Q Find
0.9614 0.0386 0 0 0 0 0.6958 0.3042 0 0 0 0 0.3083 0.6917 0 0 0 0 0.0387 0.9613 0 0 0 0 1.0000 K = 1.0000 0 0 0 0 0 0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1735 0.2600 0 0 0 0 1.0000	octave:46> Q =	step							
K = 1.0000 0 0.0383 0.6917 0 0 0 0.0387 0.9613 0 0 0 0 1.0000 K = 1.0000 0 0 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 0 0 0 1.0000	0.9614	0.0386	0	0	0				
K = 1.0000 0 0 0.0387 0.9613 0 0 0 0 1.0000 K = 1.0000 0 0 0 0 0 0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 0 1.0000 octave:47>		0.0958	0.3042	0 6017	0				
K = 1.0000 0 0 0 0 0 0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 0 0 0 1.0000	0	0	0.3083	0.0917	0 9613				
K = 1.0000 0 0 0 0 0 0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 0 1.0000 octave:47>	0	0	0	0.0507	1.0000				
1.0000 0 0 0 0 0 0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 octave:47>	К =								
0.9231 0.0769 0 0 0 0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 octave:47>	1.0000	0	0	0	0				
0.7050 0.2010 0.0940 0 0 0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 octave:47>	0.9231	0.0769	0	0	0				
0.3807 0.1857 0.1735 0.2600 0 0 0 0 1.0000 octave:47>	0.7050	0.2010	0.0940	0	0				
0 0 0 0 1.0000 octave:47>	0.3807	0.1857	0.1735	0.2600	0				
octave:47>	0	0	0	0	1.0000				
	octave:47>	•							0
twine : bash × kern : bash × kern : octave-gui ×	twine : bash	× kern :	bash × ke	ern : octave	-gui ×				

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octave:47> Q =	step							
0.9615 0 0 0	0.0385 0.6946 0 0 0	0 0.3054 0.3084 0 0	0 0.6916 0.0385 0	0 0 0.9615 1.0000				
К =								
1.0000 0.9233 0.7056 0.3812 0	0 0.0767 0.2005 0.1852 0	0 0 0.0940 0.1736 0	0 0 0.2600 0	0 0 0 1.0000				
octave:48>	•							0
twine : bash	× kern :	bash × ke	rn : octave-	-gui ×				

\geq			kern : oo	tave-gui –	- Konsole			~ ^ ×
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octave:48> Q =	step							
0.9616	0.0384	0	0	0				
0	0.6937	0.3063	0	0				
0	0	0.3085	0.6915	0				
0	0	0	0.0384	0.9616				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9234	0.0766	0	0	0				
0.7060	0.2000	0.0940	0	0				
0.3816	0.1848	0.1736	0.2600	0				
0	0	0	0	1.0000				
octave:49>	I							0
twine : bash	× kern :	bash \times ke	ern : octave	-gui ×				

			kern : o	ctave-gui –	- Konsole			
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octave:49> Q =	step							
0.9617	0.0383	0	0	0				
0	0.6931	0.3069	0	0				
0	0	0.3085	0.6915	0				
0	0	0	0.0383	0.9617				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9236	0.0764	0	0	0				
0.7063	0.1997	0.0940	0	0				
0.3819	0.1845	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:50>	•							0
twine : bash	× kern :	bash × k	ern : octave	-gui ×				

\geq			kern : o	ctave-gui –	– Konsole			
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📑 New Ta	b 🚺 Spli	it View 🗸				🗅 Сору	Paste	Q Find
octave:50> Q =	step							
0.9618	0.0382	0	0	0				
0	0.6926	0.3074	0	0				
0	0	0.3085	0.6915	0				
0	0	0	0.0382	0.9618				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9236	0.0764	0	0	0				
0.7065	0.1995	0.0940	0	0				
0.3821	0.1843	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:51>	I							0
twine : bash	× kern :	bash × ke	ern : octave	-gui ×				

$\overline{}$			kern : oo	tave-gui –	- Konsole			~ ^ ×
File Edit	View Bo	okmarks	Plugins	Settings	Help			
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octave:51> Q =	step							
0.9618	0.0382	0	0	0				
0	0.6923	0.3077	0	0				
0	0	0.3086	0.6914	0				
0	0	0	0.0382	0.9618				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9237	0.0763	0	0	0				
0.7067	0.1994	0.0940	0	0				
0.3823	0.1842	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:52>	•							0
twine : bash	× kern :	bash $ imes$ ke	ern : octave	-gui ×				

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File Edit	View Bo	ookmarks	Plugins 9	Settings	Help			
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octave:52> Q =	step							
0.9619 0 0 0	0.0381 0.6920 0 0 0	0 0.3080 0.3086 0 0	0 0.6914 0.0382 0	0 0 0.9618 1.0000				
К =								
1.0000 0.9238 0.7068 0.3824 0	0 0.0762 0.1992 0.1841 0	0 0.0940 0.1736 0	0 0 0.2599 0	0 0 0 1.0000				
octave:53>	•							0
twine : bash	× kern :	bash × ke	rn : octave	-gui ×				

			kern : oo	ctave-gui –	– Konsole			
File Edit	View Bo	ookmarks	Plugins	Settings	Help			
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octave:53> Q =	step							
0.9619	0.0381	0	0	0				
0	0.6918	0.3082	0	0				
0	0	0.3086	0.6914	0				
0	0	0	0.0381	0.9619				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9238	0.0762	0	0	0				
0.7069	0.1991	0.0940	0	0				
0.3825	0.1840	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:54>	I							0
twine : bash	× kern :	bash \times ke	ern : octave	-gui ×				

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File Edit	View Bo	ookmarks	Plugins	Settings	Help			
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octave:54> Q =	step							
0.9619	0.0381	0	0	0				
0	0.6917	0.3083	0	0				
0	0	0.3086	0.6914	0				
0	0	0	0.0381	0.9619				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9238	0.0762	0	0	0				
0.7069	0.1991	0.0940	0	0				
0.3825	0.1839	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:55>								0
twine : bash	× kern :	bash $ imes$ ke	ern : octave	-gui ×				

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octave:55> Q =	step							
0.9619	0.0381	0	0	0				
0	0.6916	0.3084	0	0				
0	0	0.3086	0.6914	0				
0	0	0	0.0381	0.9619				
0	0	0	0	1.0000				
К =								
1.0000	0	0	0	0				
0.9238	0.0762	0	0	0				
0.7070	0.1990	0.0940	0	0				
0.3826	0.1839	0.1736	0.2599	0				
0	0	0	0	1.0000				
octave:56>	I							0
twine : bash	× kern :	bash \times ke	ern : octave	-gui ×				

Instead of requiring only K(k, k + 1) = 0 for k = 0, ..., d - 1it is interesting to also require that K(k, k - 2) = 0 for k = 2, ..., d.



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In this case we find an intertwining PK = KQ where K is particularly simple.



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This construction can be repeated, leading to the intertwining we have already seen.



Note 1 This inductive construction is *not* how I proved the existence of the intertwining in the continuous-time setting.

Note 2 The eigenvalue γ_4 is the *smallest* eigenvalue, so this is *not* the Perron-Frobenius eigenvalue associated with the killed process on $\{0, \ldots, d-1\}$.

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Numerically, we have found that a known intertwining for continuous-time birth-and-death chains also holds for lazy discrete-time birth-and-death chains.

Moreover, we have numerically found a hitherto unknown intertwining for such birth-and-death chains.

This asks for a rigorous proof.

It also makes one wonder what other intertwinings wait to be discovered numerically.

I encourage everybody to try out my scripts.

However, the first indications are that outside of the world of birth-and-death chains, things may not always work so smoothly.

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The contact process

Let Λ be a finite set and let $S := \{0, 1\}^{\Lambda}$ be the set of functions $x : \Lambda \to \{0, 1\}$. For each $i, j \in \Lambda$, define $\texttt{birth}_{ij} : S \to S$ and $\texttt{dth}_i : S \to S$ by

$$\mathtt{birth}_{ij}(x)(k) := \left\{ egin{array}{cc} 1 & ext{if } x(i) = 1, \ k = j, \ x(k) & ext{otherwise} \end{array}
ight.$$

and

$$dth_i(x)(k) := \begin{cases} 0 & \text{if } k = i, \\ x(k) & \text{otherwise.} \end{cases}$$

Let p be a probability kernel on Λ and let $0 \le \lambda \le 1$. Let P be the probability kernel on S defined as

$$egin{aligned} \mathcal{P}(x,y) &:= rac{1}{|\Lambda|} \sum_{i \in \Lambda} \Big[\lambda \sum_{j \in \Lambda} \mathcal{P}(i,j) \mathbbm{1}_{\{y = \mathtt{birth}_{ij}(x)\}} \ &+ (1-\lambda) \mathbbm{1}_{\{y = \mathtt{dth}_i(x)\}} \Big]. \end{aligned}$$

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The Markov chain with transition kernel P has the following description:

- ln each step, we first choose a site *i* uniformly from Λ .
- Next, we choose to give birth with probability λ or to die with probability 1λ .
- In case of birth, we choose j according to p(i, ·) and apply birth_{ij}.
- ln case of death, we apply dth_i .
- If Λ is large, then it makes sense to rescale time by $|\Lambda|^{-1}$.

Often, there is a limit process as Λ increases to an infinite lattice: the contact process with birth rate λ and death rate $1 - \lambda$.

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On finite lattices, the Markov chain eventually gets trapped in the all-zero state $\underline{0}$.

We would like to understand how fast.

For many large lattices, it has been proved that there is a sharp transition at some 0 $<\lambda_{\rm c}<$ 1.

For $\lambda < \lambda_c$, the time till extinction is of order log $|\Lambda|$. For $\lambda > \lambda_c$, the time till extinction is of order $e^{|\Lambda|}$.

Can intertwining help us understand this better?

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Let us look at a continuous-time contact process with state space $S_m := \{0, 1\}^{\Lambda_m}$ where $\Lambda_m := \{1, 2\}^m$.

We denote the death rate by δ and assume that the birth rates $\lambda(i, j) = \lambda_{|i-j|}$ depend only on the *hierarchical distance* $|i-j| := \inf\{k : i_k \neq j_k\}$ between *i* and *j*. It is useful to picture Λ_m as the set of leaves of a binary tree.



Intertwining and coupling

Assume that *H* is intertwined on top of *G*, i.e., GK = KH. Then by **[Fill '92]** it is possible to construct a Markov process $(X_t, Y_t)_{t\geq 0}$ such that

$$\mathbb{P}\big[Y_0 \in \cdot \, \big| \, X_0\big] = \mathcal{K}(X_0, \, \cdot \,) \quad \text{a.s.}$$

implies that

$$\mathbb{P}\big[Y_t \in \cdot \, \big| \, (X_s)_{0 \leq s \leq t}\big] = \mathcal{K}(X_t, \, \cdot \,) \quad \text{a.s.} \quad (t \geq 0).$$

Moreover

$$\mathbb{P}[X_{t+\varepsilon} = x | (X_s)_{0 \le s \le t}] = \mathbb{1}(X_t, x) + \varepsilon G(X_t, x) + O(\varepsilon^2),$$

$$\mathbb{P}[Y_{t+\varepsilon} = y | (X_s, Y_s)_{0 \le s \le t}] = \mathbb{1}(Y_t, y) + \varepsilon H(Y_t, y) + O(\varepsilon^2).$$

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For each x, let H_x be a Markov generator. Assume that

$$GK = \hat{K}\hat{H},$$

where

$$\hat{\mathcal{K}}f(x):=\sum_{y}\mathcal{K}(x,y)f(x,y) \quad ext{and} \quad \hat{\mathcal{H}}f(x,y):=\sum_{y'}\mathcal{H}_{x}(y,y')f(y').$$

Then by [Athreya & S. '10] the result of [Fill '92] remains true except that now

$$\mathbb{P}\big[Y_{t+\varepsilon} = y \,\big|\, (X_s, Y_s)_{0 \le s \le t}\big] = 1(Y_t, y) + \varepsilon H_{X_t}(Y_t, y) + O(\varepsilon^2).$$

This is especially useful if H_x "does not depend too much" on x.

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Let
$$S_m := \{0, 1\}^{\Lambda_m}$$
 with $\Lambda_m := \{1, 2\}^m$.

We define a kernel K from S_m to S_{m-1} by independently replacing blocks consisting of two sites by a single site according to the following stochastic rules:

$$\begin{array}{ccc} 00\mapsto 0, & 11\mapsto 1,\\ 01 \text{ or } 10\mapsto \left\{ \begin{array}{ccc} 0 & \text{with probability } \xi,\\ 1 & \text{with probability } 1-\xi, \end{array} \right. \end{array}$$

where $\xi \in (0, \frac{1}{2}]$ is a constant, to be determined later.

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A multiscale argument



The probability of this transition is $1 \cdot (1 - \xi) \cdot \xi \cdot 1$.

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A multiscale argument

We let X be the contact process with state space S_m , birth rates $\lambda_1, \ldots, \lambda_m$, and death rate δ . We define K from S_m to S_{m-1} as described with

$$\xi := \gamma - \sqrt{\gamma^2 - \frac{1}{2}} \quad \text{with} \quad \gamma := \frac{1}{4} \left(3 + \frac{\lambda_1}{2\delta} \right).$$

Then we can construct a Markov process $(X_t, Y_t)_{t\geq 0}$ such that the generator H_x of Y in the presence of X does not depend too much on x.

In particular, Y can stochastically be estimated from below by a contact process Y' on S_{m-1} with birth rates $\lambda'_1, \ldots, \lambda'_{m-1}$ and death rate δ' , where

$$\lambda'_k := \frac{1}{2}\lambda_{k+1}$$
 and $\delta' := 2\xi\delta$.

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We may view the map

$$(\delta, \lambda_1, \ldots, \lambda_m) \mapsto (\delta', \lambda'_1, \ldots, \lambda'_{m-1})$$

as an approximate renormalisation transformation.

This can be used to derive lower bounds on the probability that the contact process survives for a long time.

Open problem Do something similar on $\Lambda = \{1, \ldots, d\}$.

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Let P be the transition kernel of the discrete-time contact process with state space $S = \{0, 1\}^2$, birth probability $0 \le \lambda \le 1$, and death probability $\delta := 1 - \lambda$.



We are looking for an intertwining of the form PK = KQ where K(00,00) = 1 and $K(x, \cdot)$ concentrated on $\{y : x \le y\}$.

This means that we choose the set Z (or its indicator Z) as follows:



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Let's try this for $\lambda = 0.6$.

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octave:2> P
P -
   1.0000
                  0
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                                     0
   0.2000 0.5000
                           0
                               0.3000
   0.2000
                  0
                      0.5000
                                0.3000
        0
            0.2000
                      0.2000
                                0.6000
octave:3> Z
Z =
 octave:4>
twine : bash \times kern : bash \times kern : octave-gui \times
```

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Let's try this for $\lambda = 0.6$.

> kern : octave-gui — Konsole								
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octave:4> Q =	step							
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К =								
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0	0	0	1.0000					
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Let's try this for $\lambda = 0.6$.

> kern : octave-gui — Konsole									~ ^ × `
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twine : ba	ash \times	kern :	bash ×	kern : octav	ve-gui $ imes$				

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What happens here is that after setting $K_0 := 1$ we can calculate $Q_1 := Q_Z(P, K_0)$ and $K_1 := K_0 + PK_0 - K_0Q_1$ all right, but when we try to calculate $Q_2 := Q_Z(P, K_1)$ we run into the problem that $C_Z(P, K_1) = \emptyset$ so there is no minimiser.

Work in progress...

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