A min-max random game on a graph that is not a tree

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Singapore, July 31, 2024

A random game

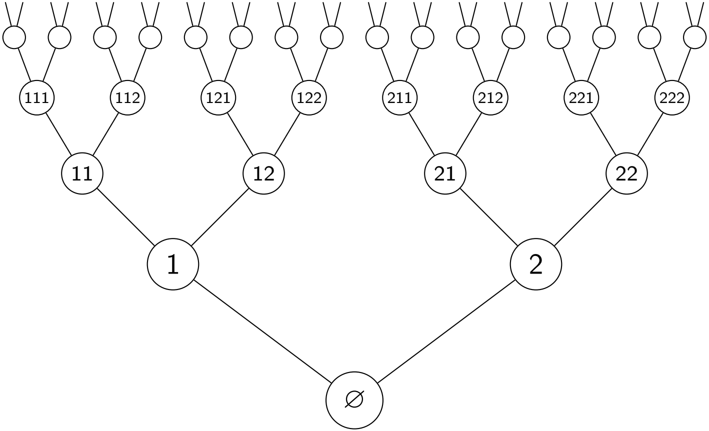
Alice and Bob play a game. They play in turn n moves each.
Alice starts and Bob plays the last move.

In each turn, Alice and Bob have two moves to choose from.
The outcome is determined by the exact sequences
of moves played by each player.
As a result, there are 2^{2n} possible outcomes of the game.

Prior to the game, we randomly assign winners to all possible outcomes in an i.i.d. way. For each possible outcome, the probability that Bob is the winner is p.

We call this game $\mathrm{AB}_n(p)$ and let $P_n^{\mathrm{AB}}(p)$ denote the probability that Bob has a winning strategy.

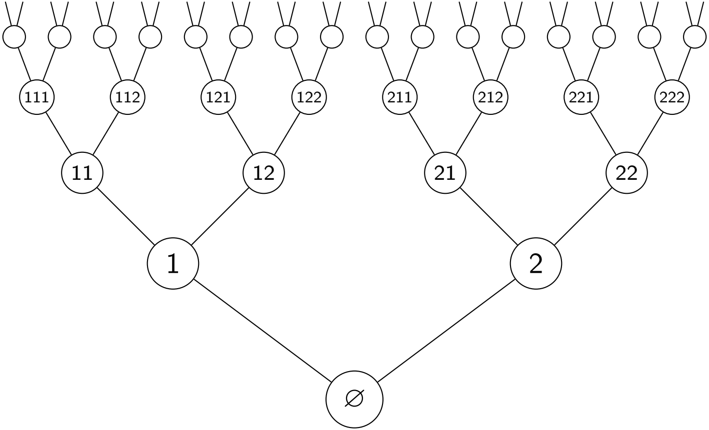
A random game



Let $\mathbb{T}$ denote the set of all finite words $\mathbf{i} = i_1 \cdots i_n$ made from the alphabet $\{1, 2\}$.

We call $|\mathbf{i}| := n$ the *length* of the word $\mathbf{i} = i_1 \cdots i_n$

A random game

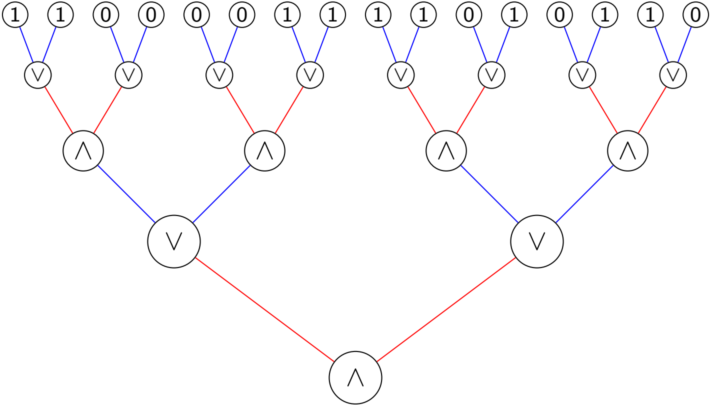


We write $[\mathbb{T}]_n := \{\mathbf{i} \in \mathbb{T} : |\mathbf{i}| \leq n\}$,

$\langle\mathbb{T}\rangle_n := \{\mathbf{i} \in \mathbb{T} : |\mathbf{i}| < n\}$,

and $\partial_n \mathbb{T} := \{\mathbf{i} \in \mathbb{T} : |\mathbf{i}| = n\}$.

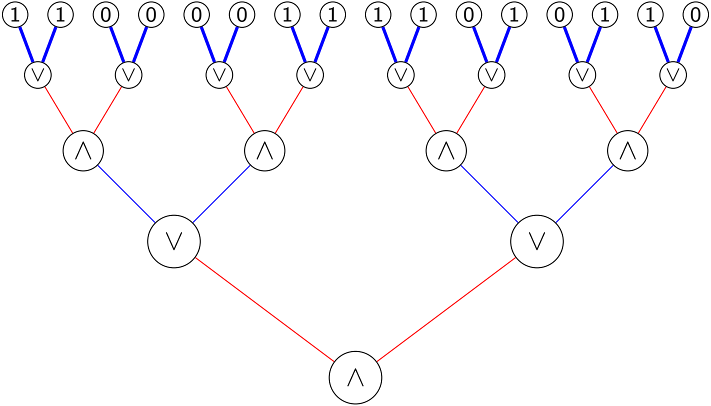
A random game



Let $\big(X(\mathbf{i})\big)_{\mathbf{i}\in\partial_{2n}\mathbb{T}}$ be i.i.d. $\{0,1\}$-valued random variables with $\mathbb{P}\big[X(\mathbf{i})=1\big]=p$.

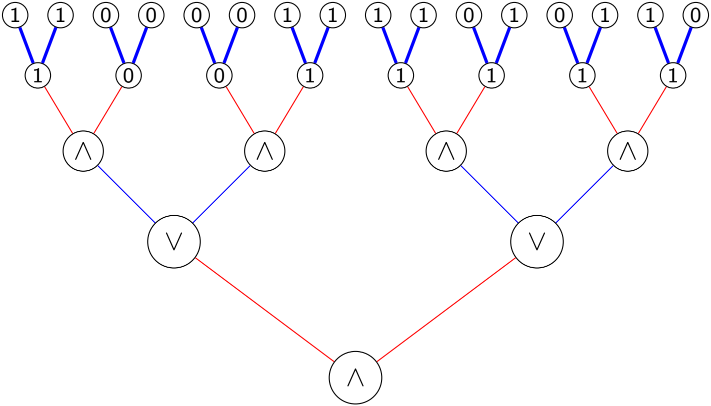
A 0 means a win for Alice and a 1 a win for Bob.

A random game



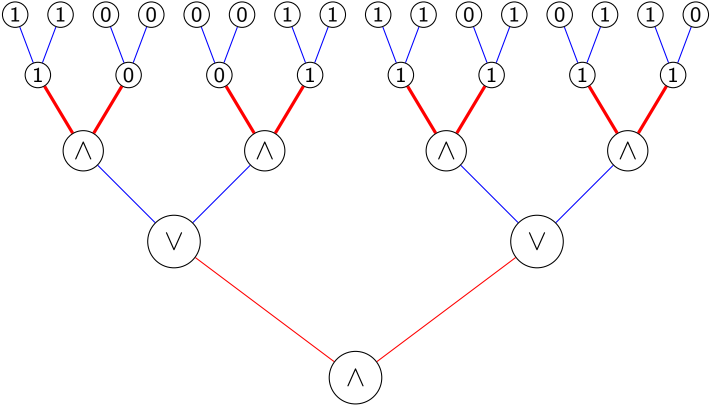
Bob has the last move and chooses
the maximum of $X(\mathbf{i}1)$ and $X(\mathbf{i}2)$.
We set $\overline{X}(\mathbf{i}) := X(\mathbf{i})$ $(\mathbf{i} \in \partial_{2n}\mathbb{T})$.

A random game



Bob has the last move and chooses
the maximum of $X(\mathbf{i}1)$ and $X(\mathbf{i}2)$.
We define $\overline{X}(\mathbf{i}) := \overline{X}(\mathbf{i}1) \vee \overline{X}(\mathbf{i}1)$ for $\mathbf{i} \in \partial_k \mathbb{T}$ with k odd.

A random game



In her last move Alice chooses
the minimum of $X(\mathbf{i}1)$ and $X(\mathbf{i}2)$.
We define $\overline{X}(\mathbf{i}) := \overline{X}(\mathbf{i}1) \wedge \overline{X}(\mathbf{i}1)$ for $\mathbf{i} \in \partial_k \mathbb{T}$ with k even.

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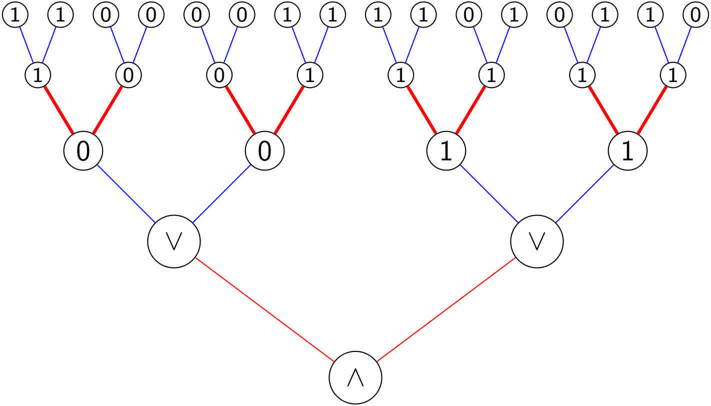

In her last move Alice chooses
the minimum of $X(\mathbf{i}1)$ and $X(\mathbf{i}2)$.
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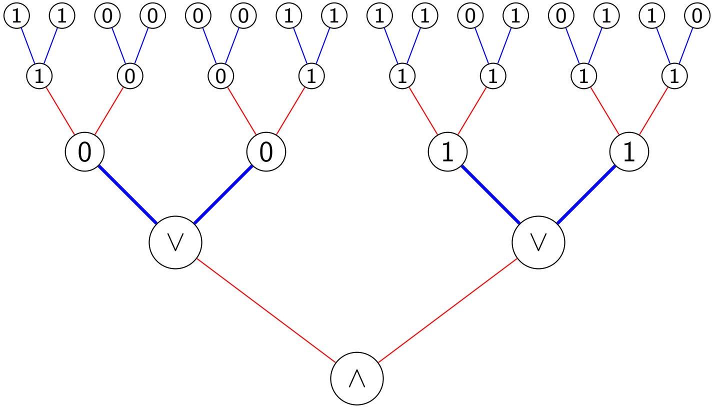

In the move before that, Bob chooses the maximum again.

$\overline{X}(\mathbf{i}) := \overline{X}(\mathbf{i}1) \vee \overline{X}(\mathbf{i}1)$ for $\mathbf{i} \in \partial_k \mathbb{T}$ with k odd.

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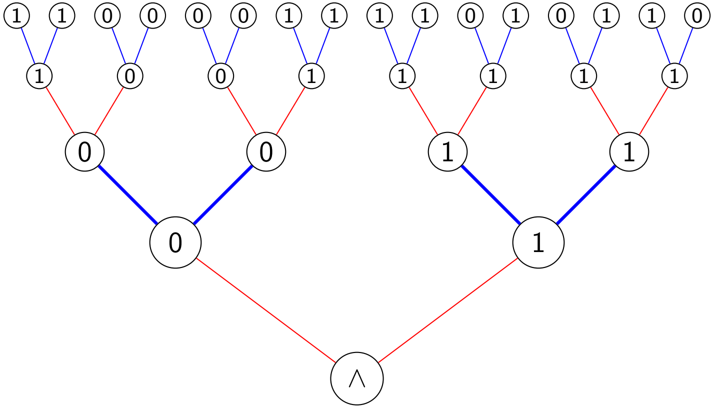

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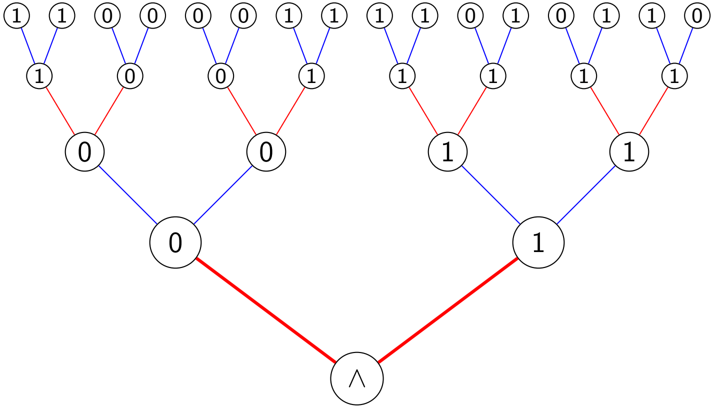
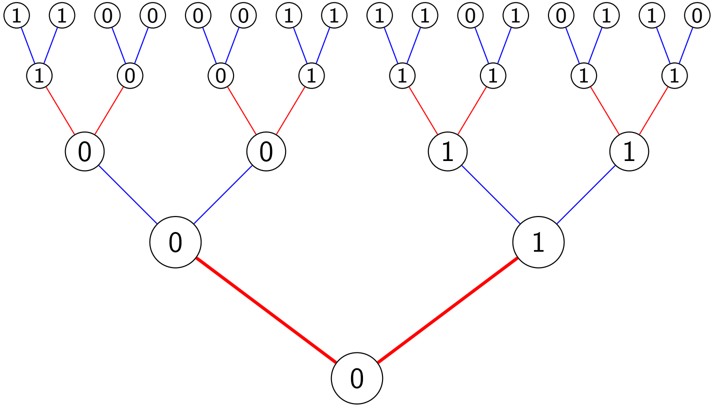


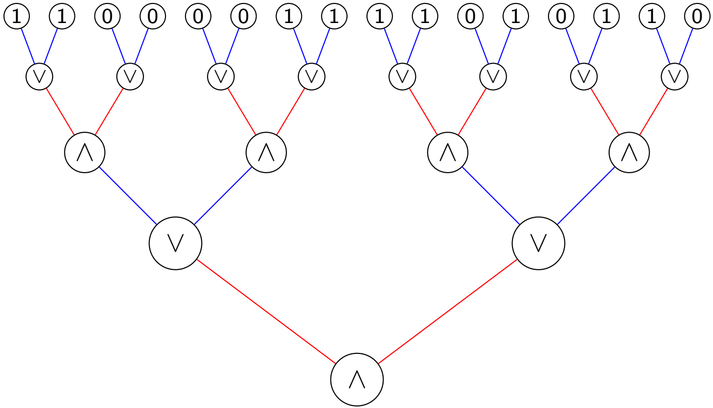
And in her first move, Alice chooses the minimum.
$\overline{X}(\mathbf{i}) := \overline{X}(\mathbf{i}1) \wedge \overline{X}(\mathbf{i}1)$ for $\mathbf{i} \in \partial_k \mathbb{T}$ with k odd.

A random game



$X(\varnothing) = 0$, so in this game, Alice has a winning strategy.

A random game



These sort of *minmax trees* or *game trees* have long been used in game theory.
The idea to use i.i.d. input is due to Judea Pearl (1980).

A random game

$P_n^{\mathrm{AB}}(p)$ denotes the probability that Bob has a winning strategy.

Pearl (1980) proved that

$$P_n^{\mathrm{AB}}(p) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p < p_{\mathrm{c}}^{\mathrm{AB}}, \\ p_{\mathrm{c}}^{\mathrm{AB}} & \text{if } p = p_{\mathrm{c}}^{\mathrm{AB}}, \\ 1 & \text{if } p > p_{\mathrm{c}}^{\mathrm{AB}}, \end{cases}$$

where $p_{\mathrm{c}}^{\mathrm{AB}} := \frac{1}{2}(3-\sqrt{5}) \approx 0.382$, which has the effect that $p_{\mathrm{c}}^{\mathrm{AB}} : 1 - p_{\mathrm{c}}^{\mathrm{AB}}$ is the golden ratio.

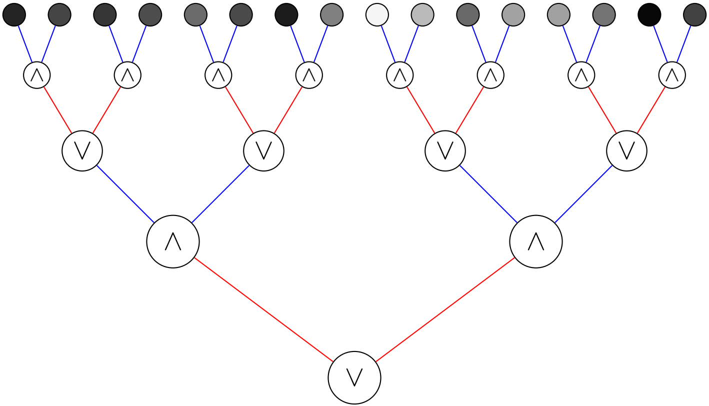
Note that $p_{\mathrm{c}}^{\mathrm{AB}} < 1/2$, which is due to the fact that Bob has the last move, which gives him an advantage.

Continuous game-values

In a different variant of the game, prior to the game, we assign i.i.d. $\mathrm{Unif}[0,1]$ distributed random variables $\big(U(\mathbf{i})\big)_{\mathbf{i}\in\partial_{2n}\mathbb{T}}$ to the possible outcomes of the game.

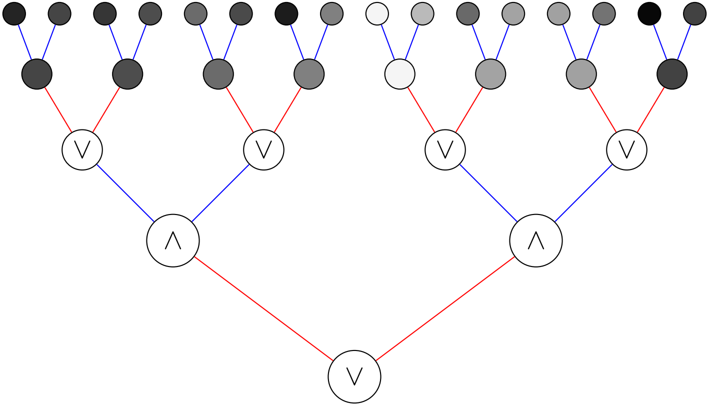
If the game ends in the outcome $\mathbf{i}$, then the pay-out for Alice $U(\mathbf{i})$ and for Bob is $1-U(\mathbf{i})$.

Continuous game-values



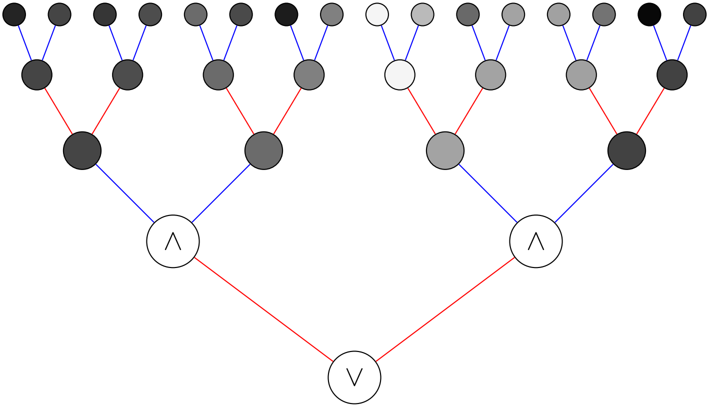
In this game, Bob chooses the *minimum*.

Continuous game-values



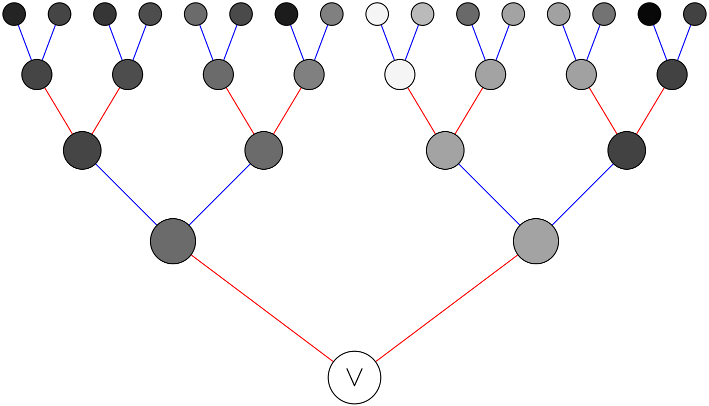
In this game, Bob chooses the *minimum*.

Continuous game-values



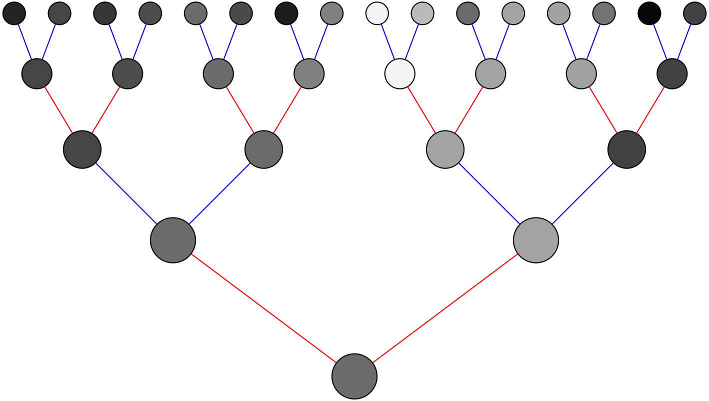
And Alice chooses the maximum.

Continuous game-values



So we set $\overline{U}(\mathbf{i}) := U(\mathbf{i})$ $(\mathbf{i} \in \partial_{2n}\mathbb{T})$

Continuous game-values



And define $\overline{U}(\mathbf{i}) := \begin{cases} \overline{U}(\mathbf{i}1) \wedge \overline{U}(\mathbf{i}2) & \text{for } \mathbf{i} \in \partial_k \mathbb{T} \text{ with } k \text{ odd,} \\ \overline{U}(\mathbf{i}1) \vee \overline{U}(\mathbf{i}2) & \text{for } \mathbf{i} \in \partial_k \mathbb{T} \text{ with } k \text{ even} \end{cases}$

Continuous game-values

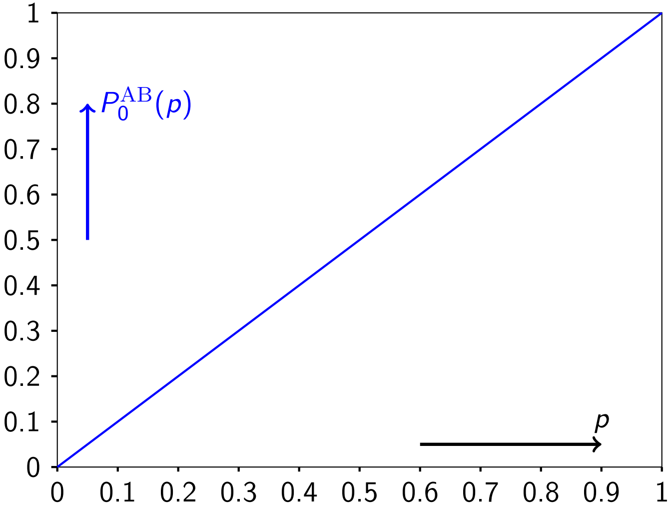
If we set $\overline{X}(\mathbf{i}) := \begin{cases} 0 & \text{if } \overline{U}(\mathbf{i}) > p, \\ 1 & \text{if } \overline{U}(\mathbf{i}) \leq p, \end{cases}$

then we obtain the same $\big(\overline{X}(\mathbf{i})\big)_{\mathbf{i}\in[\mathbb{T}]_n}$ as before.

The yields a *coupling* of processes with different values of p.

$$P_n^{\mathrm{AB}}(p) = \mathbb{P}\big[\overline{U}(\varnothing) \leq p\big] \qquad \big(p \in [0,1]\big).$$

A random game



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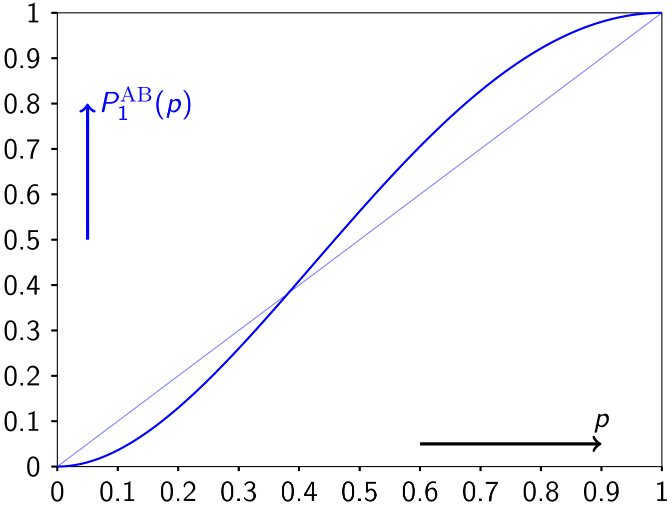

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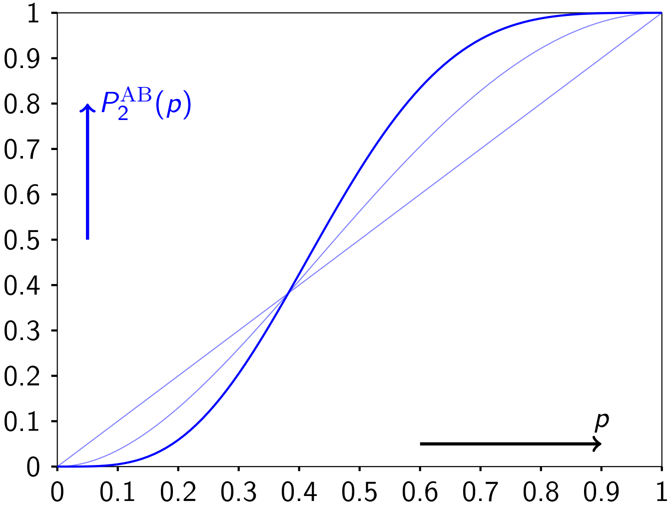

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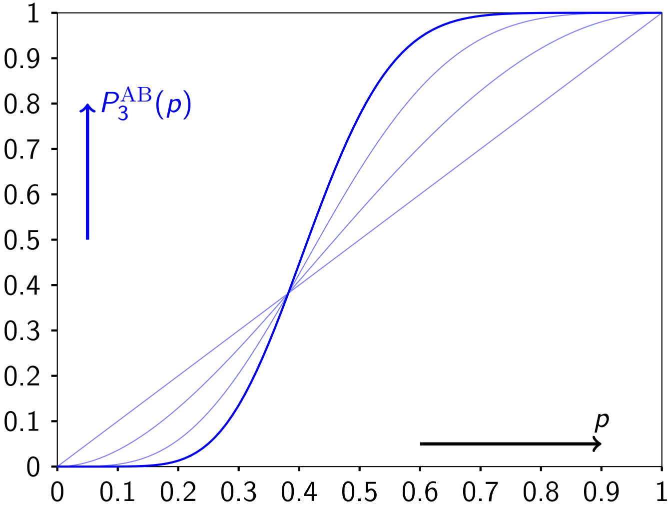

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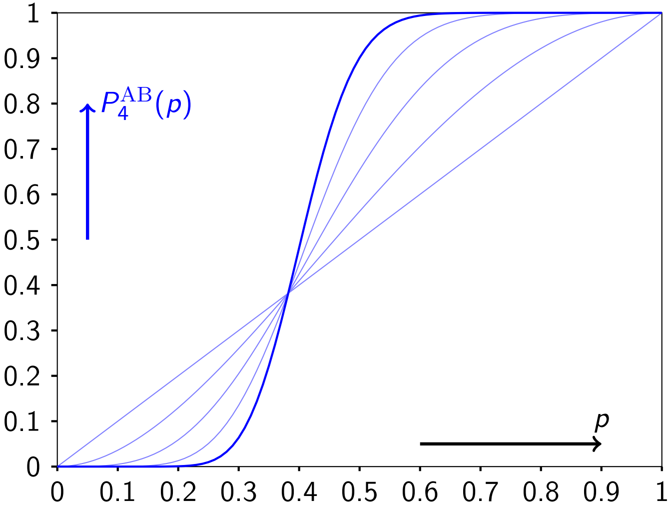

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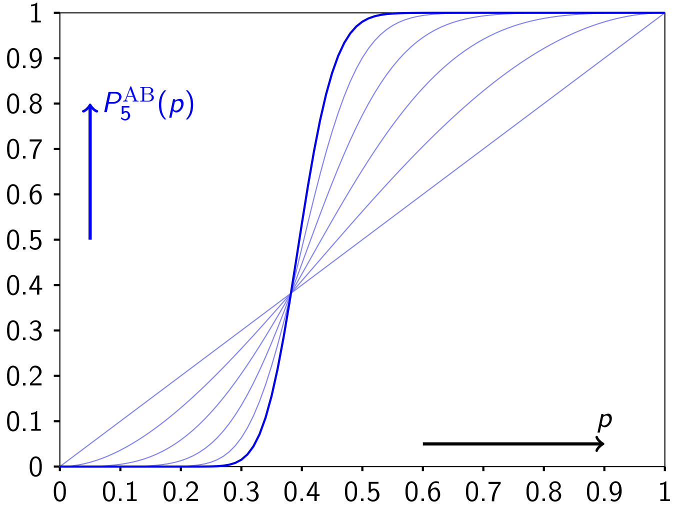

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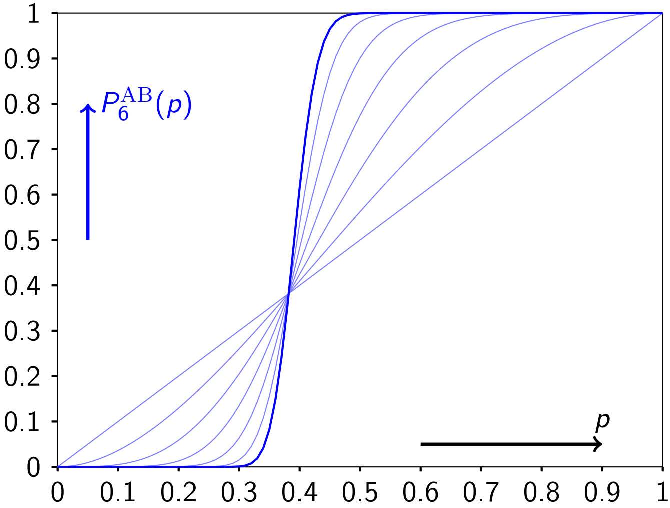

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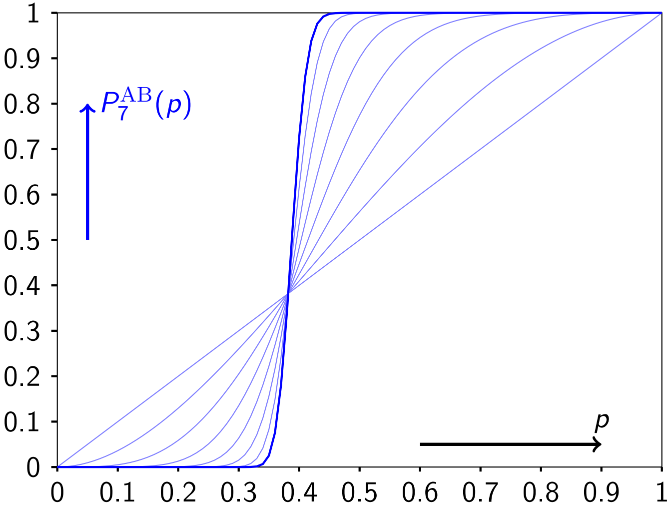

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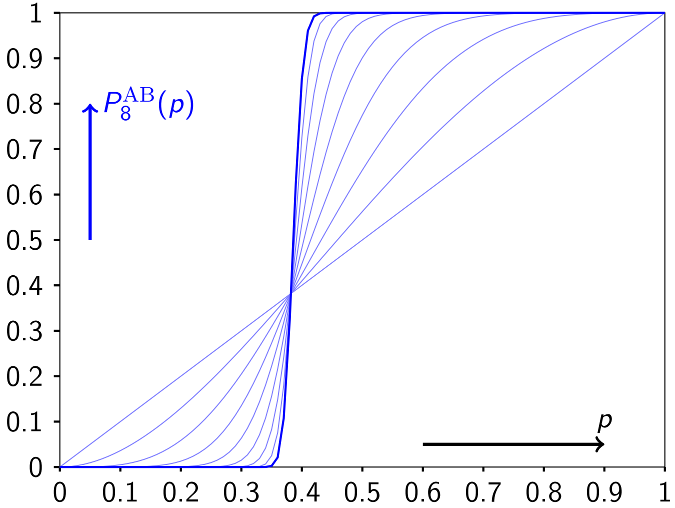

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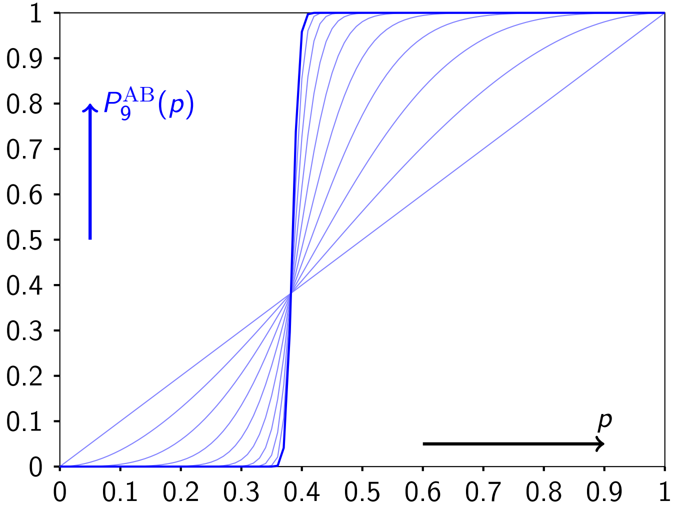

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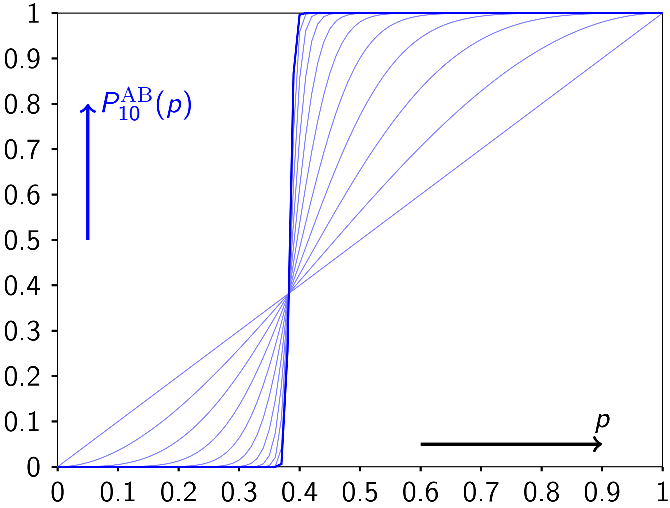


A random game

Ali Kahn, Devroye, and Neininger (2005) have proved that for a suitable choice of $0 < \xi < 1$,

$$P_n^{\mathrm{AB}}(p_{\mathrm{c}} + \xi^n q) \underset{n\to\infty}{\longrightarrow} F(q) \qquad (q \in \mathbb{R}),$$

for some nontrivial distribution function $F : \mathbb{R} \to [0,1]$.

Several variants of Pearl's game have been studied where the deterministic tree is replaced by a random tree, such as a Galton-Watson tree, or a tree where for each internal node randomness decides whose turn it is.

More general game-graphs

What if the "game tree" is not a tree?

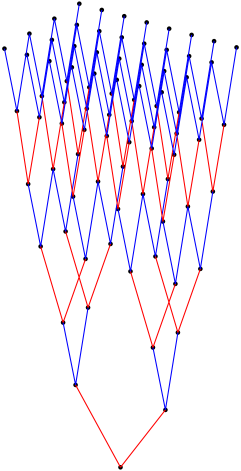
What if different game histories can lead to the same outcome?

More general game-graphs

Let $\mathrm{Ab}_n(p)$ denote the modified game where the outcome is determined by the exact sequence of moves played by Alice as before, but for Bob all that matters is how often he has played each of the two possible moves.

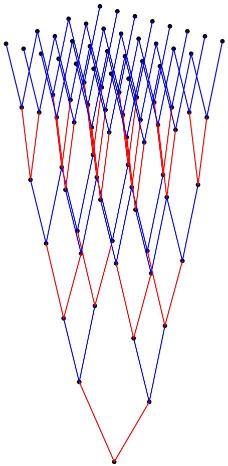
In this case, there are $2^n \cdot (n+1)$ possible outcomes to which we assign winners in an i.i.d. way as before.

More general game-graphs



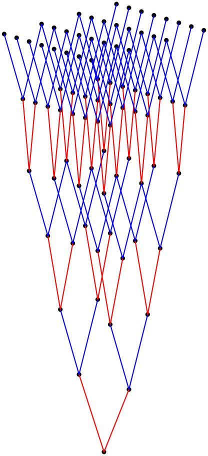
In this case, the game-graph is not a tree.

More general game-graphs



The game-graph of the game $\mathrm{Ab}_3(p)$.

More general game-graphs



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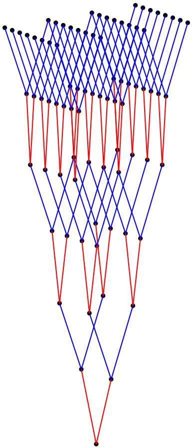

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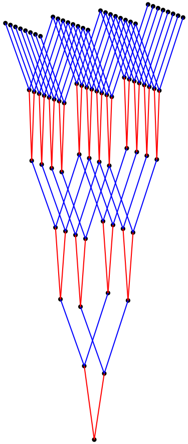

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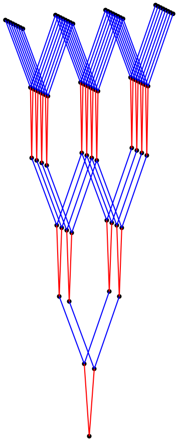

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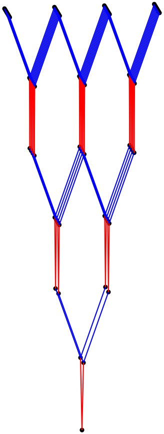

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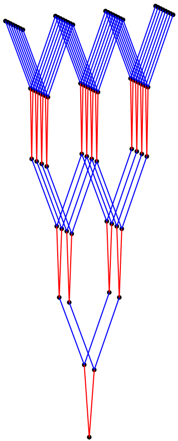

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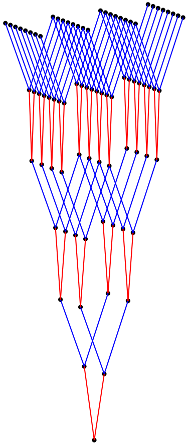

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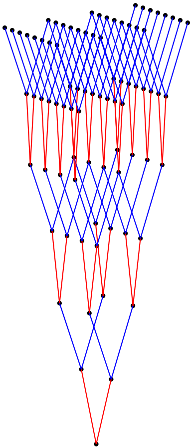

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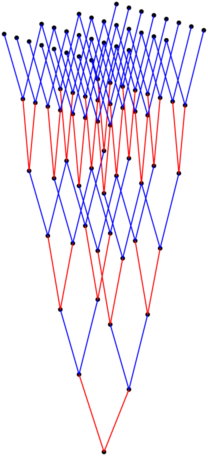

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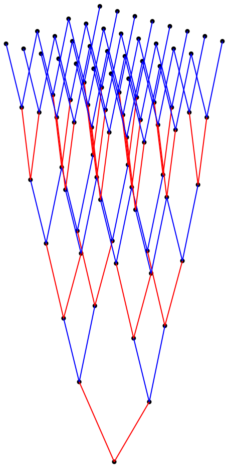

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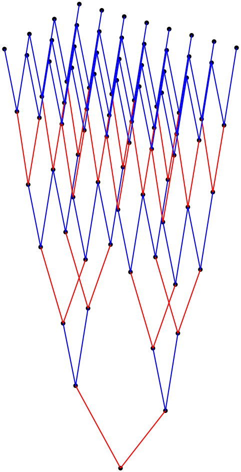

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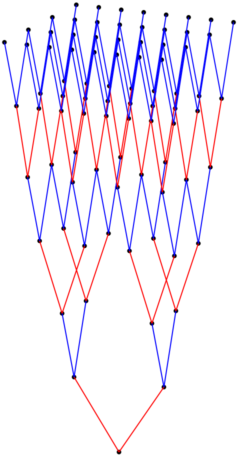

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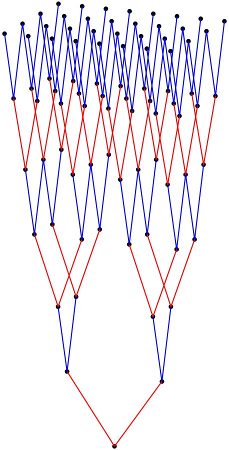

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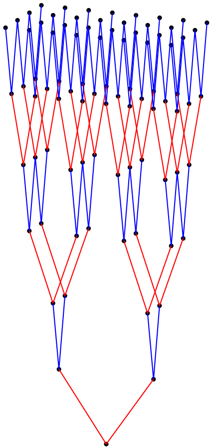

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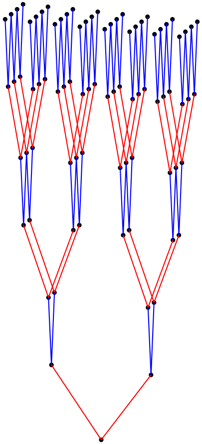

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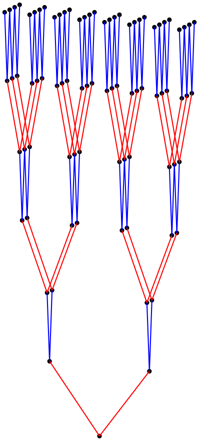

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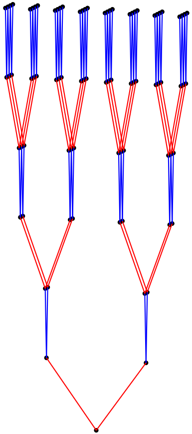

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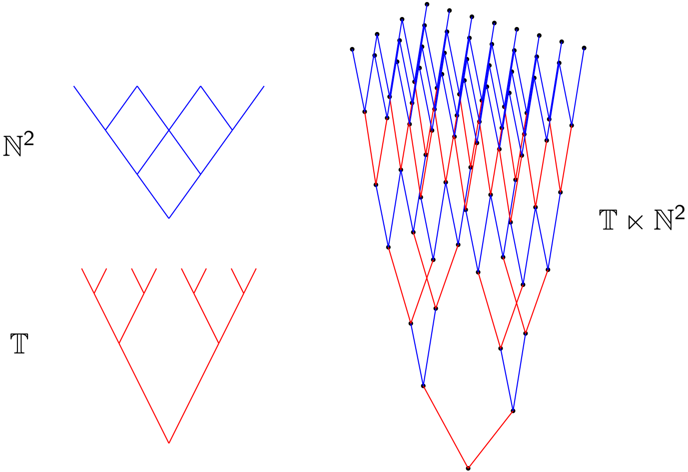


We can view the game-graph as a sort of “product” $\mathbb{T} \ltimes \mathbb{N}^2$ of graphs $\mathbb{T}$ and $\mathbb{N}^2$ describing the moves of the individual players.

More general game-graphs

In this way, we can define four different games:

$\mathrm{AB}_n(p)$ Pearl's original game with game-graph $\mathbb{T} \ltimes \mathbb{T} \cong \mathbb{T}$.

$\mathrm{Ab}_n(p)$ The game with game-graph $\mathbb{T} \ltimes \mathbb{N}^2$.

$\mathrm{aB}_n(p)$ The game with game-graph $\mathbb{N}^2 \ltimes \mathbb{T}$.

$\mathrm{ab}_n(p)$ The game with game-graph $\mathbb{N}^2 \ltimes \mathbb{N}^2$.

We let $P_n^{\mathrm{AB}}(p)$, $P_n^{\mathrm{Ab}}(p)$, $P_n^{\mathrm{aB}}(p)$, and $P_n^{\mathrm{ab}}(p)$ denote the probability that Bob has a winning strategy.

[Sturm, Cardona-Tobón & S. '24] One has

$$P_n^{\mathrm{Ab}}(p) \leq P_n^{\mathrm{AB}}(p) \leq P_n^{\mathrm{aB}}(p) \qquad \big(p \in [0,1],\ n \geq 1\big).$$

Conjecture

$$P_n^{\mathrm{Ab}}(p) \leq P_n^{\mathrm{ab}}(p) \leq P_n^{\mathrm{aB}}(p) \qquad \big(p \in [0,1],\ n \geq 1\big).$$

More general game-graphs

[Sturm, Cardona-Tobón & S. '24] There exist constants $0 < p_c^{\mathrm{Ab}}, p_c^{\mathrm{aB}} < 1$ such that

$$P_n^{\mathrm{Ab}}(p) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if and only if } p < p_c^{\mathrm{Ab}}, \\ 1 & \text{if } p > p_c^{\mathrm{Ab}}, \end{cases}$$

$$P_n^{\mathrm{aB}}(p) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p < p_c^{\mathrm{aB}}, \\ 1 & \text{if and only if } p > p_c^{\mathrm{aB}}. \end{cases}$$

One has $1/2 \leq p_c^{\mathrm{Ab}} \leq 7/8$ and $1/16 \leq p_c^{\mathrm{aB}} \leq \frac{1}{2}(3-\sqrt{5})$.

Note that $P_n^{\mathrm{AB}}(p) \leq P_n^{\mathrm{aB}}(p)$ implies $p_c^{\mathrm{aB}} \leq p_c^{\mathrm{AB}} = \frac{1}{2}(3-\sqrt{5}) \approx 0.382$.

More general game-graphs

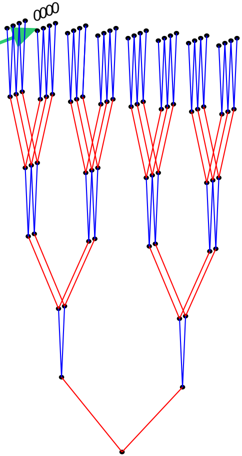
For the game $\mathrm{ab}_n(p)$, we can only prove that

$$P_n^{\mathrm{ab}}(p) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p < 1/64, \\ 1 & \text{if } p > 15/16. \end{cases}$$

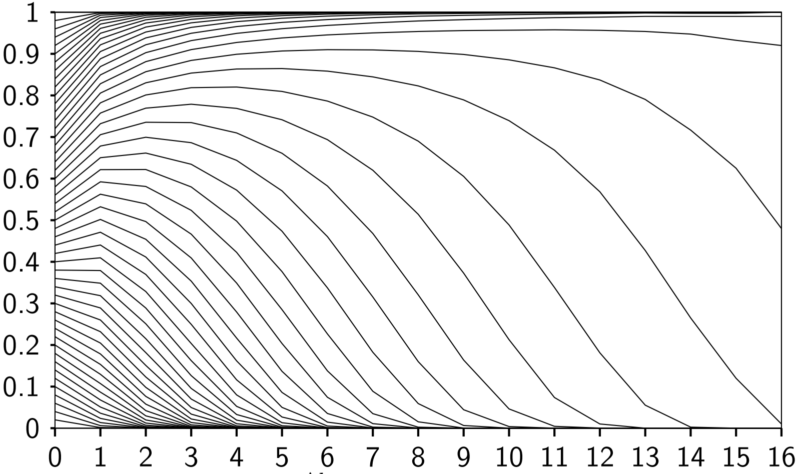
More general game-graphs

The bound $1/2 \leq p_{\mathrm{c}}^{\mathrm{Ab}}$ follows from the fact that if $p < 1/2$, then for large n, with high probability, at least one of these sets contains only zeros.

This means that Alice has a winning strategy that does not even react to Bob's moves.

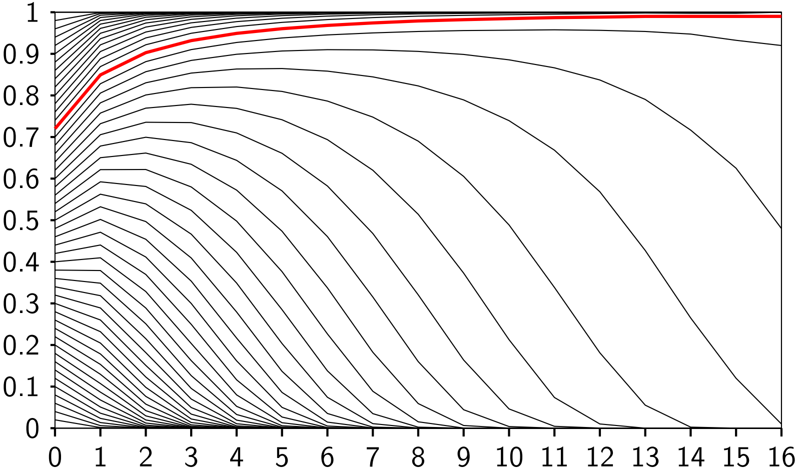


More general game-graphs



Numerical data suggest $p_c^{\mathrm{Ab}} \approx 0.72$
and $\lim_{n\to\infty} P_n^{\mathrm{Ab}}(p_c^{\mathrm{Ab}})$ is one or close to one.

More general game-graphs



Numerical data suggest $p_{\rm c}^{\rm Ab} \approx 0.72$
and $\lim_{n\to\infty} P_n^{\rm Ab}(p_{\rm c}^{\rm Ab})$ is one or close to one.

A cellular automaton

Let $U_0 = \big(U_0(i,j)\big)_{(i,j)\in\mathbb{N}^2}$ be i.i.d. $\mathrm{Unif}[0,1]$ distributed.

For each of the combinations $\mathrm{xx} = \mathrm{AB}$, Ab, aB, and ab, we can define $(U_t^{\mathrm{xx}})_{t\geq 0}$ by $U_0^{\mathrm{xx}} := U_0$ and

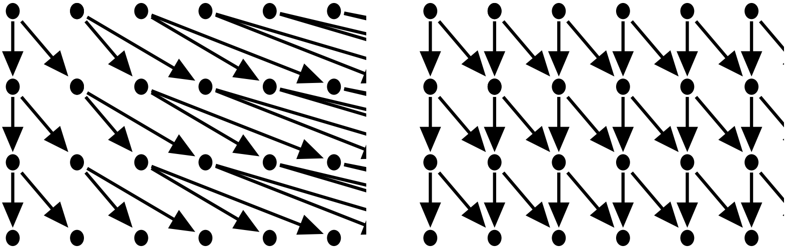
$$\left.\begin{array}{ll} \mathrm{A.} & U_{t+1}^{\mathrm{xx}}(i,j) = U_t^{\mathrm{xx}}(2i,j) \vee U_t^{\mathrm{xx}}(2i+1,j), \\ \mathrm{a.} & U_{t+1}^{\mathrm{xx}}(i,j) = U_t^{\mathrm{xx}}(i,j) \vee U_t^{\mathrm{xx}}(i+1,j) \end{array}\right\} \quad \text{if } t \text{ is odd,}$$

$$\left.\begin{array}{l} \mathrm{B.}\ U_{t+1}^{\mathrm{xx}}(i,j) = U_t^{\mathrm{xx}}(i,2j) \wedge U_t^{\mathrm{xx}}(i,2j+1), \\ \mathrm{b.}\ U_{t+1}^{\mathrm{xx}}(i,j) = U_t^{\mathrm{xx}}(i,j) \wedge U_t^{\mathrm{xx}}(i,j+1) \end{array}\right\} \quad \text{if } t \text{ is even.}$$

We claim that

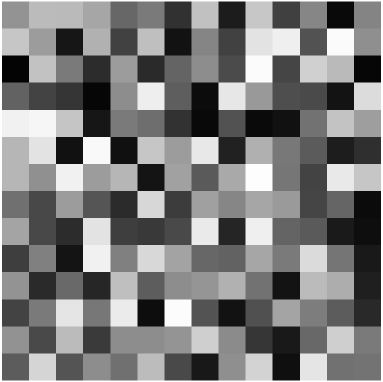
$$P_n^{\mathrm{xx}}(p) = \mathbb{P}\big[U_{2n}^{\mathrm{xx}}(0,0) \leq p\big].$$

A cellular automaton



We observe that $U^{\mathrm{Ab}}_{2n}(0,0)$ depends on $\big(U_0(i,j)\big)_{(i,j)\in\mathbb{N}^2}$ exactly in the way $\overline{U}(\varnothing)$ depends on $\big(U(v)\big)_{v\in\partial_{2n}(\mathbb{T}\ltimes\mathbb{N}^2)}$.

A cellular automaton

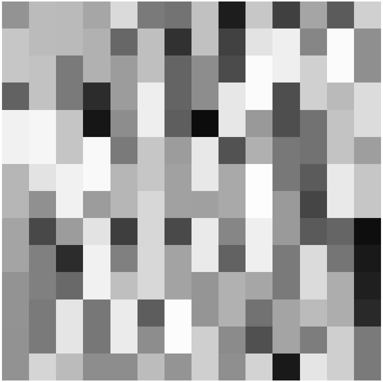


$t = 0$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

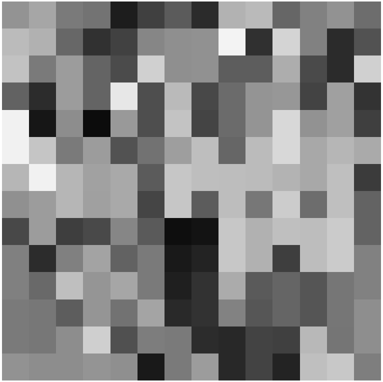
A cellular automaton



$t = 1$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

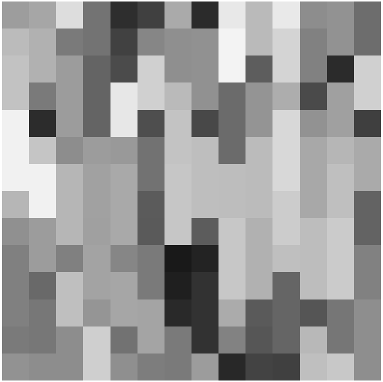
A cellular automaton



$t = 2$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

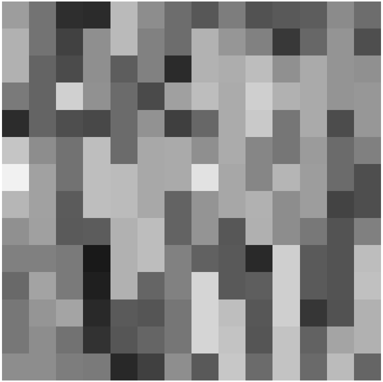
A cellular automaton



$t = 3$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

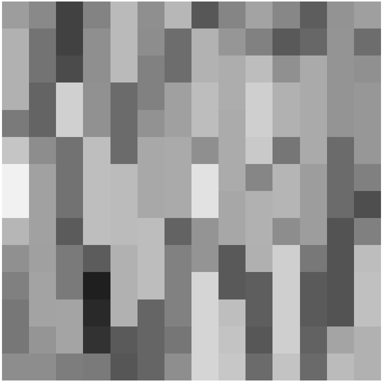


$t = 4$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

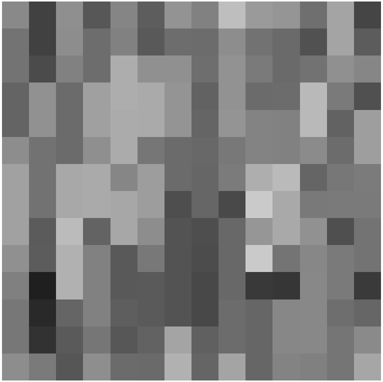


$t = 5$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

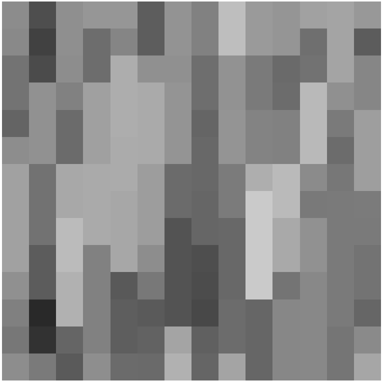
A cellular automaton



$t = 6$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

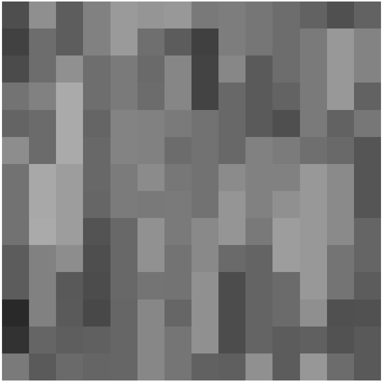


$t = 7$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

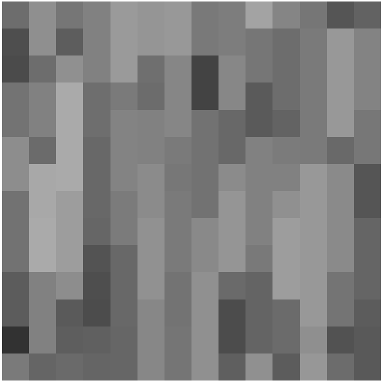
A cellular automaton



$t = 8$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

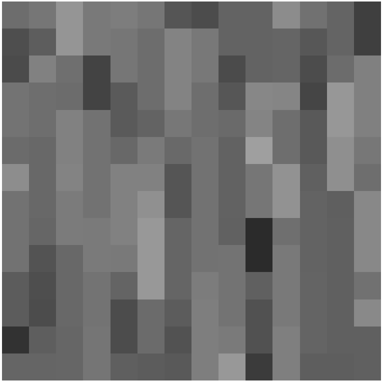


$t = 9$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

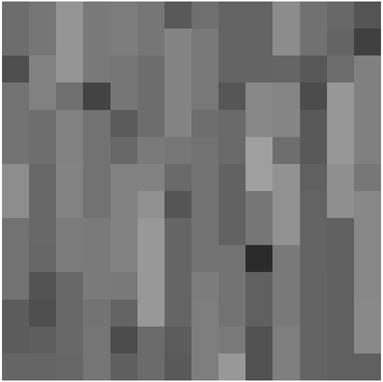
A cellular automaton



$t = 10$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

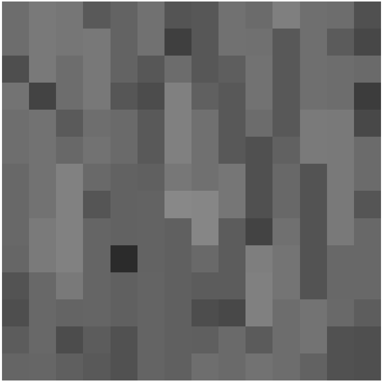


$t = 11$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Grayscales indicate a value between zero (white) and one (black).

A cellular automaton

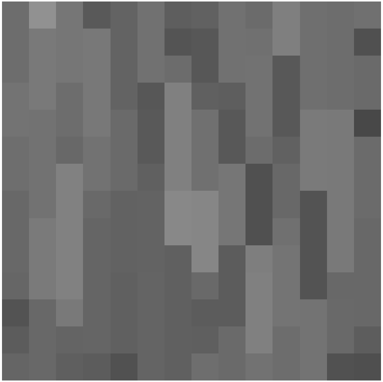


$t = 12$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Columns are independent of each other at all times.

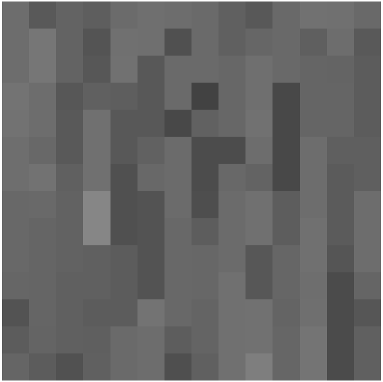
A cellular automaton



$t = 13$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Columns are independent of each other at all times.

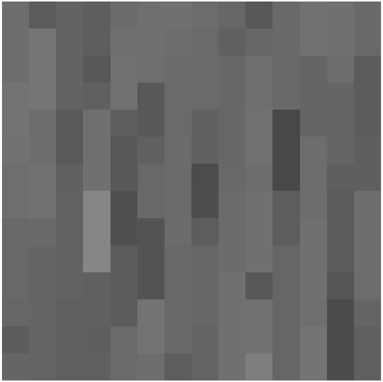
A cellular automaton



$t = 14$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Columns are independent of each other at all times.

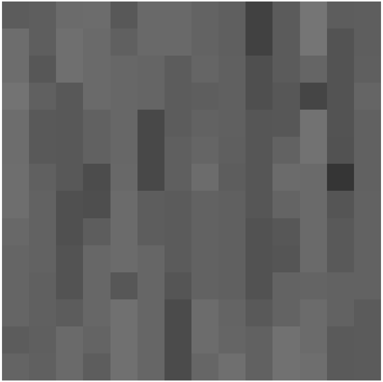
A cellular automaton



$t = 15$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

A cellular automaton

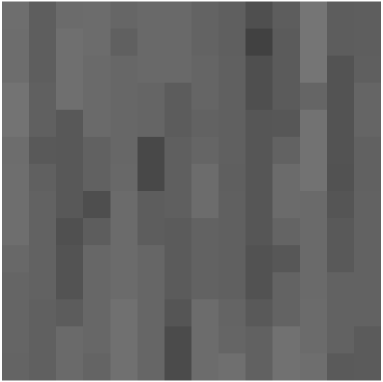


$t = 16$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Columns are independent of each other at all times.

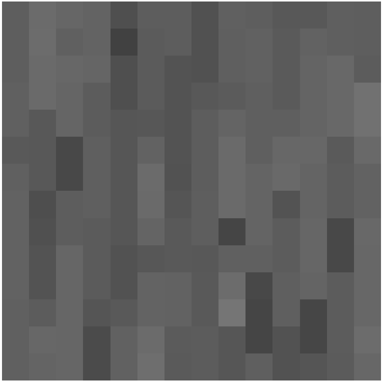
A cellular automaton



$t = 17$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

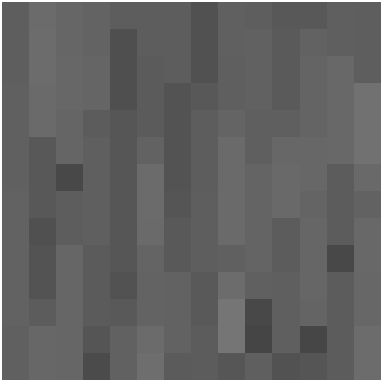
A cellular automaton



$t = 18$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

A cellular automaton

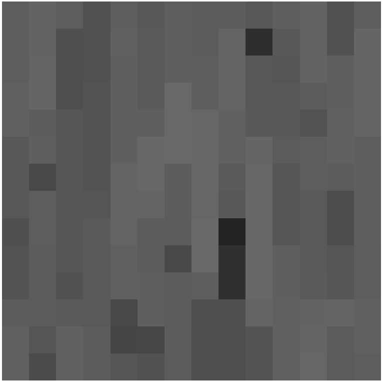


$t = 19$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Columns are independent of each other at all times.

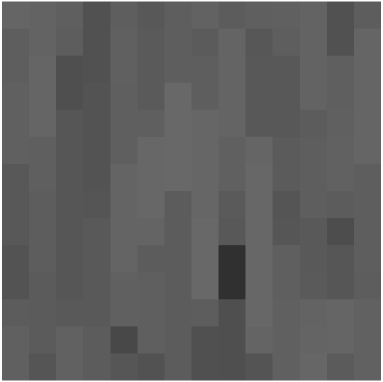
A cellular automaton



$t = 20$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

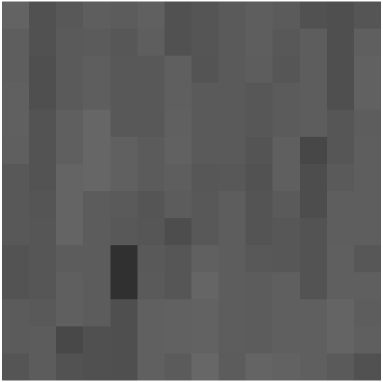
A cellular automaton



$t = 21$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

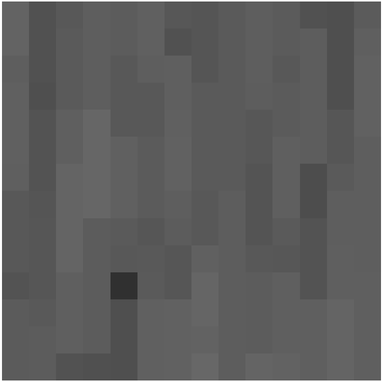
A cellular automaton



$t = 22$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t \geq 0}$.
Columns are independent of each other at all times.

A cellular automaton

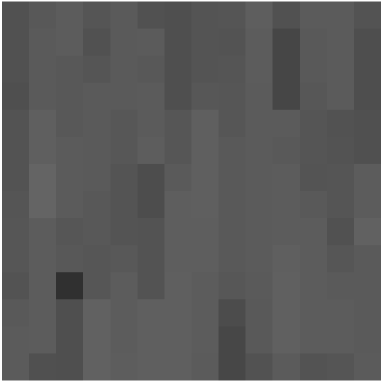


$t = 23$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.

Columns are independent of each other at all times.

A cellular automaton



$t = 24$

The cellular automaton $(U_t^{\mathrm{Ab}})_{t\geq 0}$.
Columns are independent of each other at all times.

A cellular automaton

For the cellular automaton $(U_t^{\mathrm{aB}})_{t\geq 0}$, rows are independent of each other at all times.

For the cellular automaton $(U_t^{\mathrm{AB}})_{t\geq 0}$, all lattice points remain independent of each other at all times.

Bounds on the critical values

To prove the bounds $p_{\rm c}^{\rm Ab} \leq 7/8$ and $1/16 \leq p_{\rm c}^{\rm aB}$, as well as the fact that

$$P_n^{\rm ab}(p) \underset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } p < 1/64, \\ 1 & \text{if } p > 15/16, \end{cases}$$

we use a Peierls argument due to Toom (1980) and further developed by S., Szábo, and Toninelli (2024).

Strategies

A *strategy* for Alice (Bob) is a function that assigns to each state in which it is Alice's (Bob's) turn precisely one of the two moves available to Alice (Bob). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the set of strategies for Alice and Bob, respectively, and let

$$o(\sigma_1, \sigma_2)$$

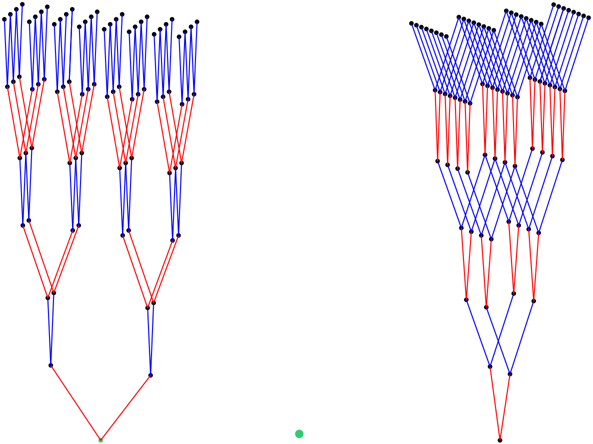
denote the outcome of the game if Alice plays strategy $\sigma_1 \in \mathcal{S}_1$ and Bob plays strategy $\sigma_2 \in \mathcal{S}_2$. We set

$$Z(\sigma_1) := \big\{ o(\sigma_1, \sigma_2) : \sigma_2 \in \mathcal{S}_2 \big\}.$$

A strategy $\sigma_1 \in \mathcal{S}_1$ is *winning* for Alice if

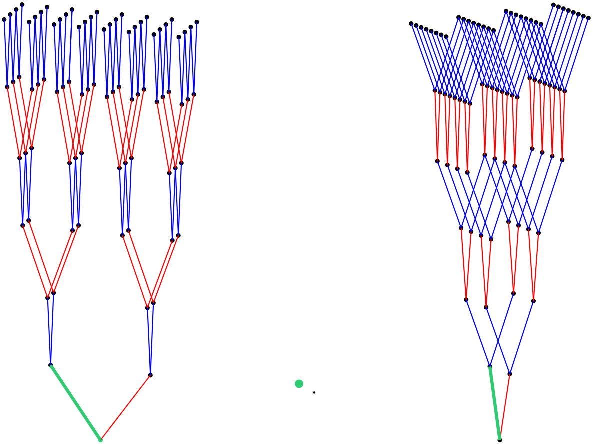
$$X(v) = 0 \quad \forall v \in Z(\sigma_1).$$

Strategies



Construction of the set $Z(\sigma_1)$ for a given strategy of Alice.

Strategies



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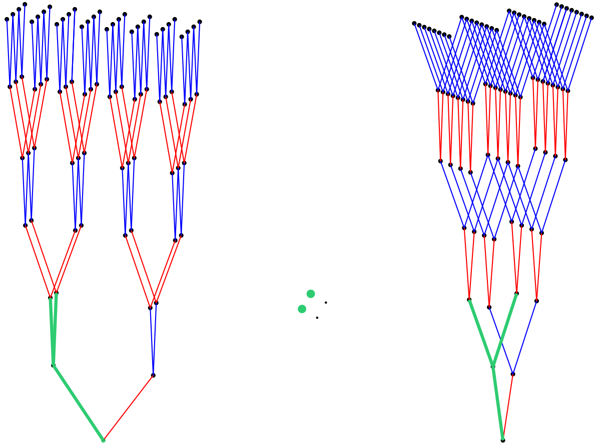

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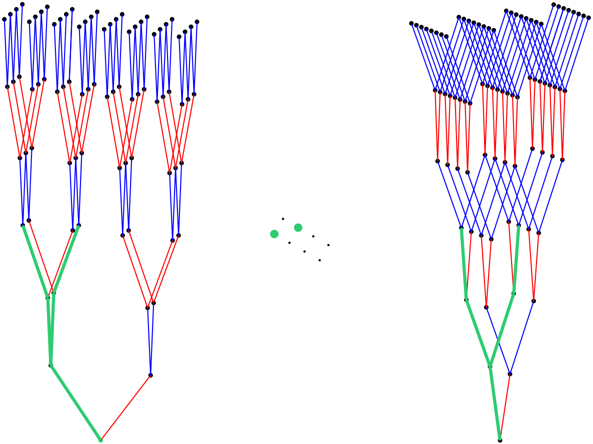
Strategies



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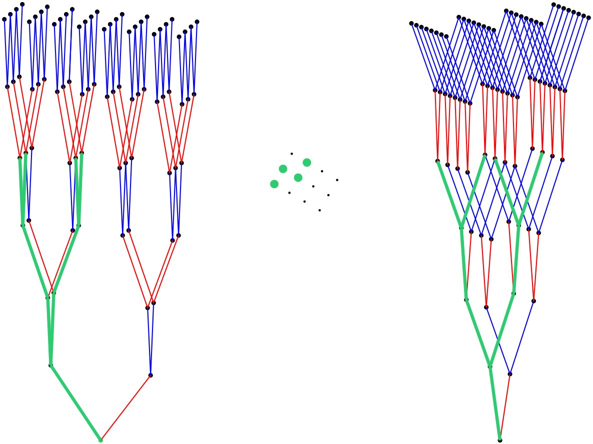
Strategies



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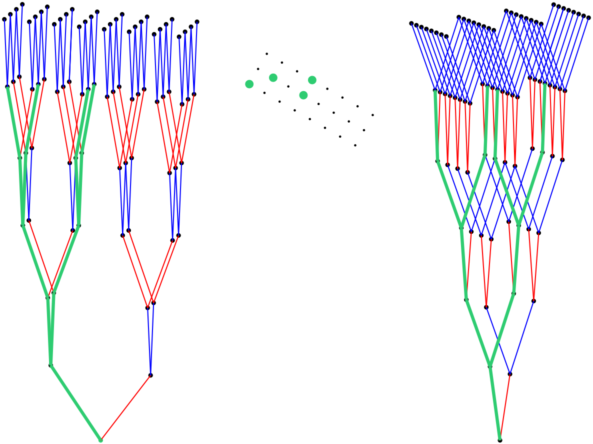
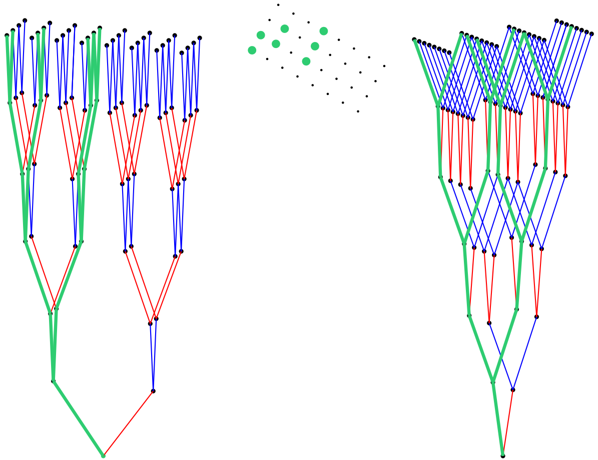
Strategies



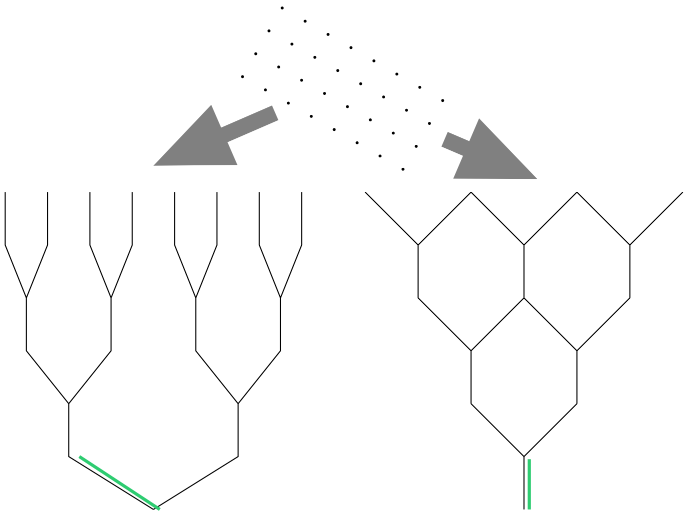
Construction of the set $Z(\sigma_1)$ for a given strategy of Alice.

Strategies



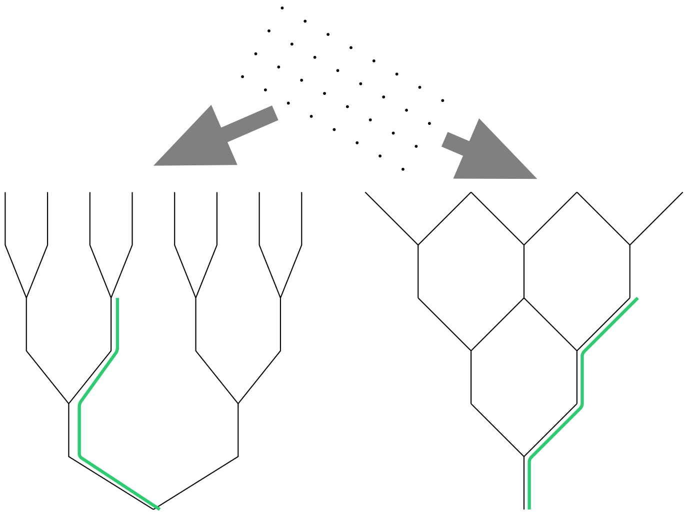
Construction of the set $Z(\sigma_1)$ for a given strategy of Alice.

Toom cycles



We will construct a *Toom cycle* that passes through $Z(\sigma_1)$.

Toom cycles



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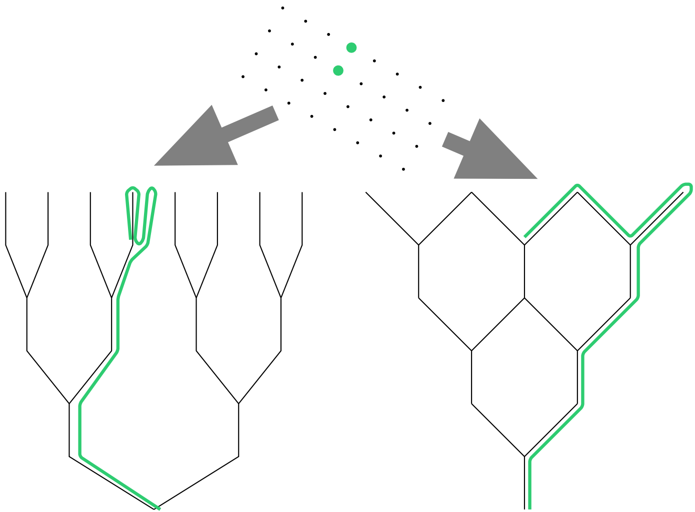

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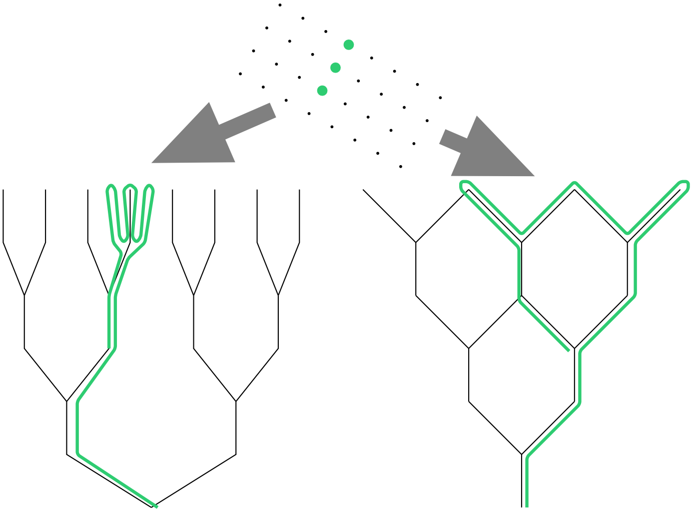

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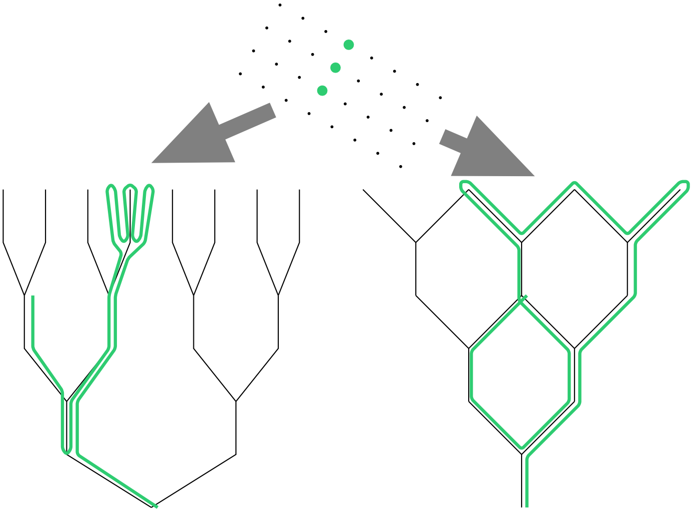

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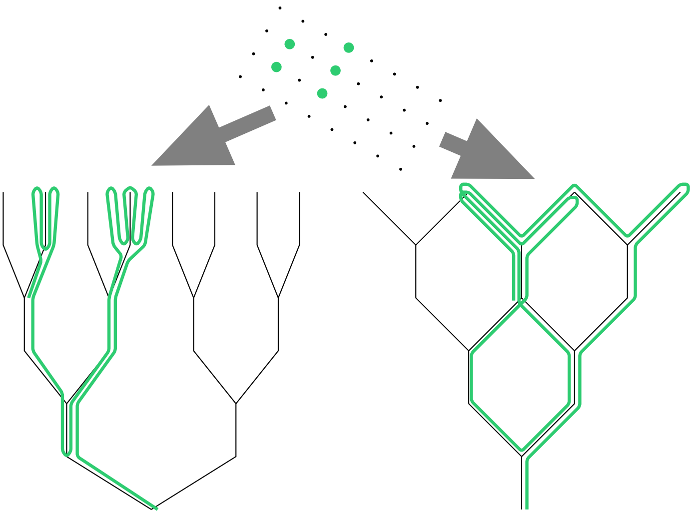
Toom cycles



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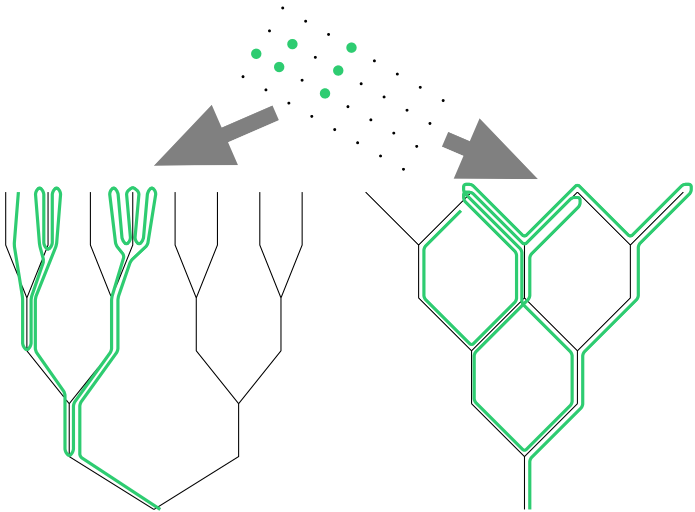
Toom cycles



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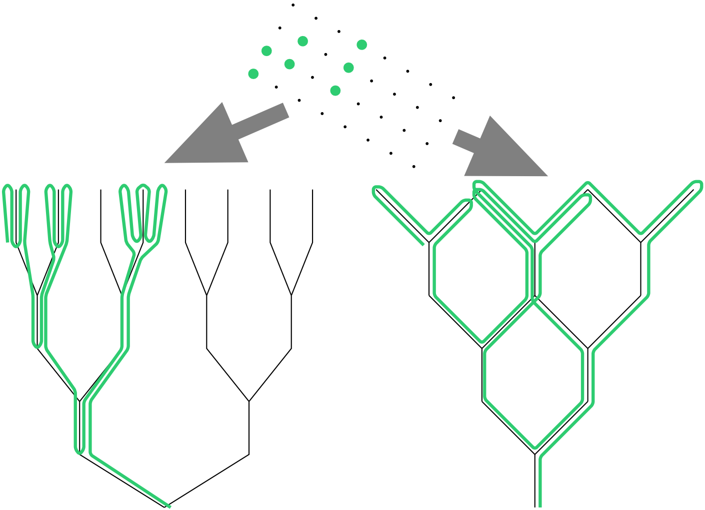
Toom cycles



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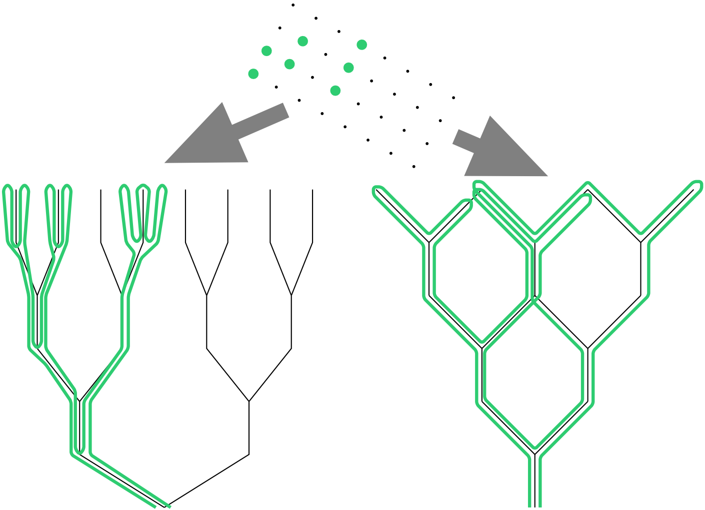
Toom cycles



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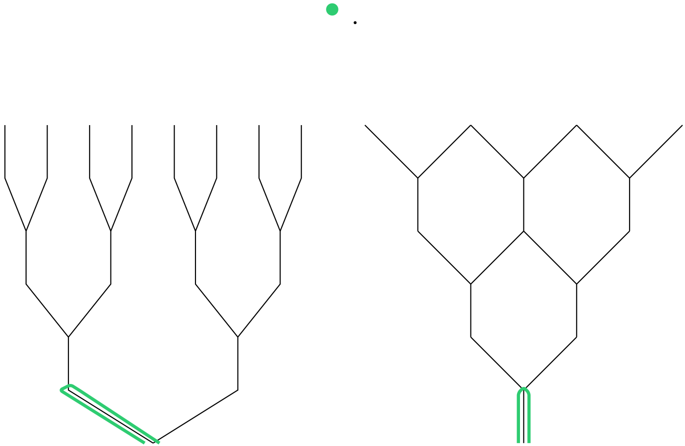
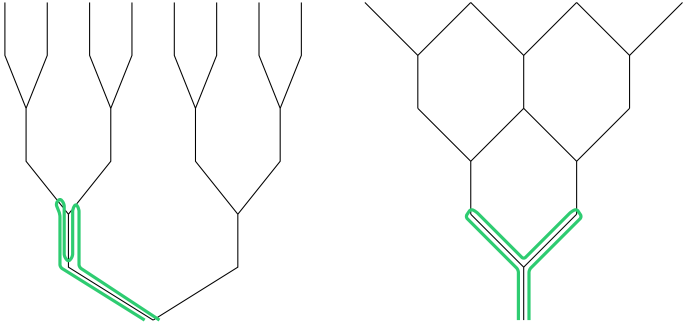
Toom cycles



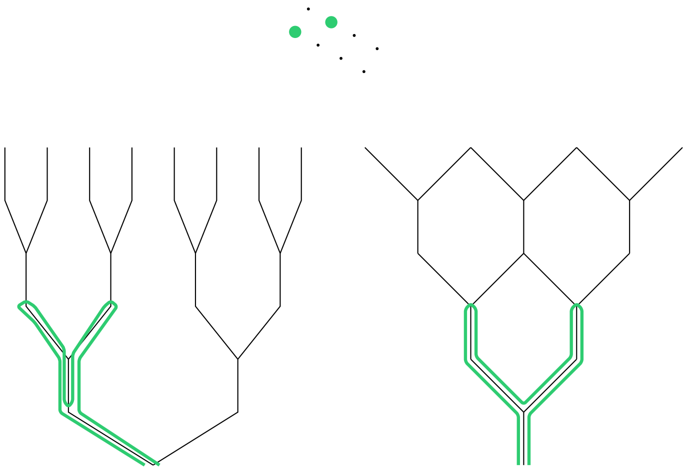
The construction is by induction and uses *loop erasion*.

Toom cycles



The construction is by induction and uses *loop erasion*.

Toom cycles



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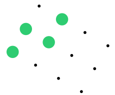

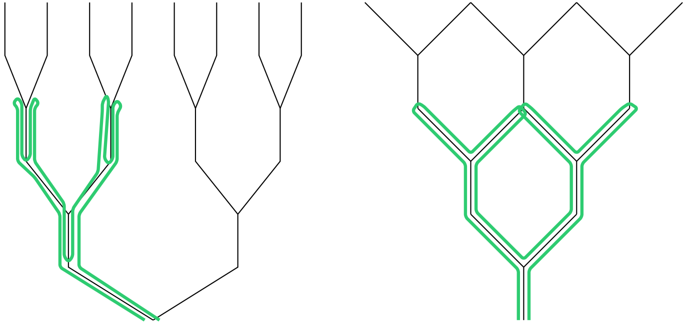

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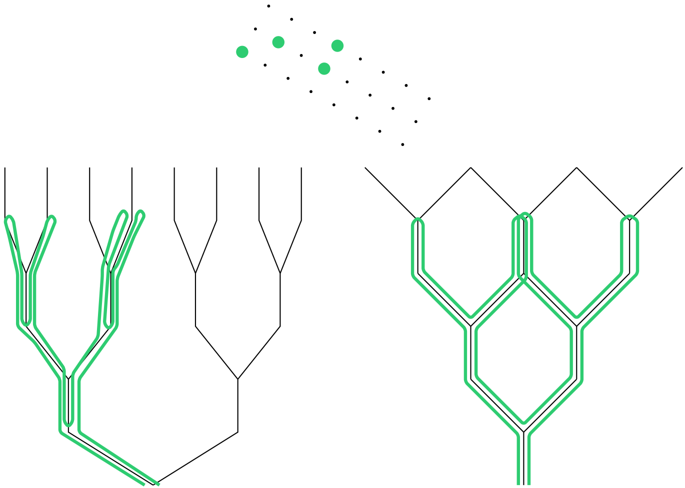
Toom cycles



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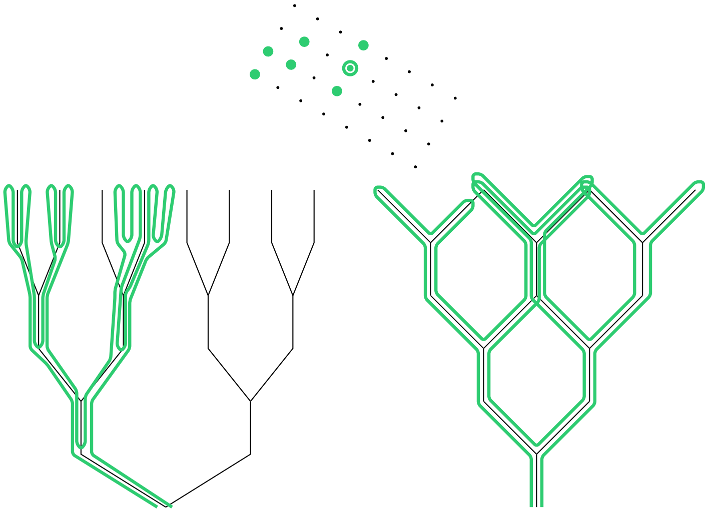

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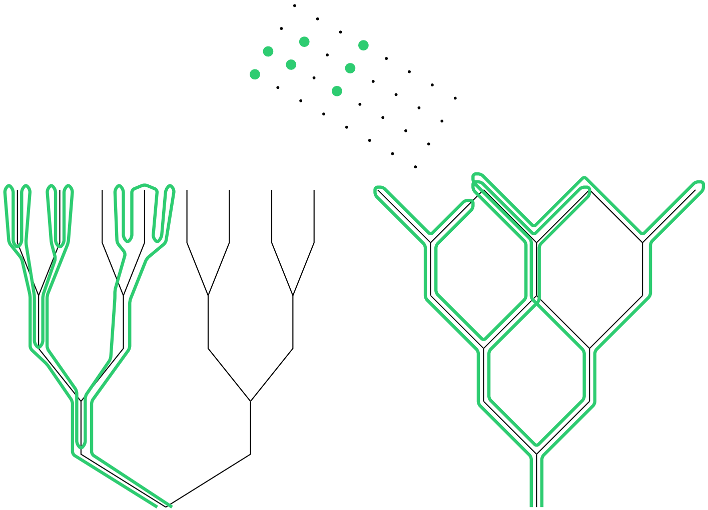


The construction is by induction and uses *loop erasion*.

Toom cycles

Theorem If Alice has a winning strategy, then there exists a Toom cycle ψ such that $X(v) = 0$ for each possible outcome v that ψ passes through.

Lemma For each Toom cycle ψ, there exists an integer $m \geq 0$ such that the cycle makes m steps in each of the six directions

straight-up, straight-down, right-up, right-down, left-up, and left-down,

and ψ passes through $m + 1$ possible outcomes.

Lemma For each m, there are $\leq 8^m$ different Toom cycles.

Consequence:

$$1 - P_n^{\mathrm{Ab}}(p) \leq \sum_{m=n}^{\infty} 8^m (1 - p)^{m+1},$$

and $p_{\mathrm{c}}^{\mathrm{Ab}} \leq 7/8$.

Sharpness of the transition

Let S be a finite set, let $L : \{0,1\}^S \to \{0,1\}$ be a function, and let $X^p = \big(X^p(v)\big)_{v\in S}$ be i.i.d. with $\mathbb{P}[X^p(v) = 1] = p$ $(v \in S)$. Let

$$X^p_{v,y}(w) := \begin{cases} y & \text{if } w = v, \\ X^p(w) & \text{otherwise,} \end{cases} \qquad (y = 0, 1).$$

We say v is *pivotal* if $L(X^p_{v,0}) \neq L(X^p_{v,1})$. The *influence* of v is

$$I^p(v) := \mathbb{P}\big[v \text{ is pivotal in } X^p\big] \qquad (v \in S).$$

If L is monotone, then *Russo's formula* says that

$$\tfrac{\partial}{\partial p}\mathbb{P}\big[L(X^p) = 1\big] = \sum_{v\in S} I^p(v).$$

Sharpness of the transition

Bourgain, Kahn, Kalai, Katznelson, and Linial (1992) have proved that there exists a universal constant $c > 0$ such that:

$$(\star) \quad \sum_{v \in S} I^p(v) \geq c \mathrm{Var}\big(L(X^p)\big) \log\big(1 / \sup_{v \in S} I^p(v)\big).$$

If each individual influence is small, and the law of $L(X^p)$ is nontrivial, then the sum of the influences must be large.

We apply this to $S := \partial_n(\mathbb{T} \ltimes \mathbb{N}^2)$ and $L_n(x) := 1$ iff Bob has a winning strategy for $\big(x(v)\big)_{v \in \partial_n(\mathbb{T} \ltimes \mathbb{N}^2)}$.

We observe that $\mathrm{Var}\big(L_n(X^p)\big) = P_n^{\mathrm{Ab}}(p)\big(1 - P_n^{\mathrm{Ab}}(p)\big)$.

Sharpness of the transition

Assume $\varepsilon \leq P_n^{\rm Ab}(p) \leq 1-\varepsilon$.
Formula $(\star)$ tells us that

$$(\star) \quad \sum_{v\in S} I_n^p(v) \geq c\varepsilon(1-\varepsilon)\log(1/J_n)$$

with $J_n := \sup_{v\in S} I_n^p(v)$.

Because of the symmetry of $\partial_n(\mathbb{T}\ltimes\mathbb{N}^2)$,

$$\#\{v\in\partial_n(\mathbb{T}\ltimes\mathbb{N}^2) : I_n^p(v) = J_n\} \geq 2^n.$$

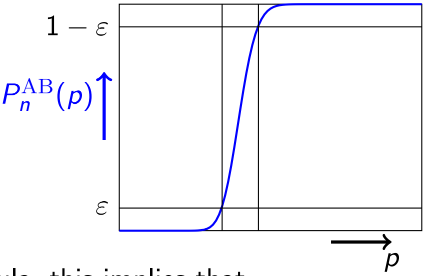
As a consequence,

$$\sum_{v\in S} I_n^p(v) \geq 2^n J_n.$$

Combining this with $(\star)$ one finds that for some $c' > 0$

$$\sum_{v\in S} I_n^p(v) \geq c'\varepsilon(1-\varepsilon)n \quad \text{if } \varepsilon \leq P_n^{\rm Ab}(p) \leq 1-\varepsilon.$$

Sharpness of the transition



By Russo's formula, this implies that

$$\frac{\partial}{\partial p} P_n^{\mathrm{Ab}}(p) \geq c' \varepsilon (1-\varepsilon) n \quad \text{if } \varepsilon \leq P_n^{\mathrm{Ab}}(p) \leq 1-\varepsilon,$$

which implies that $P_n^{\mathrm{Ab}}(p)$ increases from a value $\leq \varepsilon$ to a value $\geq 1-\varepsilon$ in an interval of length $\leq 1/(c'\varepsilon(1-\varepsilon)n)$.

Sharpness of the transition.