

Available themes for a PhD student (combinations possible)

I. THE SKELETAL PROCESS OF BRANCHING BROWNIAN MOTIONS

Consider a continuous-time spatial branching process, where particles move according to independent Brownian motions with drift $-\sqrt{2}$ and are absorbed at the origin, and moreover, with branching rate β , particles split into two new particles, created on the position of the parent. It is known that this system exhibits a phase transition in the sense that for $\beta \leq 1$, eventually all particles are absorbed in the origin, while for $\beta > 1$, the process survives on the positive halfline with positive probability. In a sequence of papers [BBS11, BBS13, BBS14, BBS15], J. Berestycki, N. Berestycki, and J. Schweinsberg have studied the near-critical limit $\beta \downarrow 1$. Their main result is that there exists a time scale T_β such that the genealogy of a finite sample of particles, collected at time T_β , converges to the genealogy of Neveu's branching process [Nev92].

The *skeletal process* of a branching process is the process obtained by throwing away all particles whose offspring will eventually die out. It is known that skeletal processes are themselves branching processes, but with a different offspring distribution from the original process. The skeletal process of spatial branching processes in general also has a different underlying motion [FS04]. Skeletal processes have proved to be a valuable tool in the study of spatial branching processes; see [EKW15] for an overview. This leads to the following problem:

Problem 1 *Show that as $\beta \downarrow 1$, on time scales of order T_β , the skeletal process of near-critical branching Brownian motions converges to the skeletal process of Neveu's branching process.*

This problem can probably be solved using the results in [BBS11, BBS13, BBS14, BBS15], but these papers are long and technical. A more promising and interesting approach is to use methods from [FS04, EKW15]. In particular, it seems it should be possible to use the theory of intertwining of Markov processes [Kur98, Swa13] and in particular the methods developed in [KR11] to distinguish the particles of branching Brownian motion into different types that each capture different aspects of the behaviour. This approach is of wider interest in the theory of metastability.

II. COOPERATIVE BRANCHING

Particle systems with cooperative branching are prototypes of interacting particle systems that are monotone, but not additive. They arise naturally in a biological context [DFL86, Nob92, Neu94, SS14] and also in bootstrap percolation [MT19]. The early paper [Dur85] announced some remarkable claims about a two-dimensional system, the proofs of which have unfortunately never appeared.

As with any interacting particle system, the first step towards understanding spatial models is understanding the mean-field model, which corresponds to a well-mixing population with no spatial structure. In the mean-field model, there is a striking difference between the contact process and systems with cooperative branching and deaths. The contact process has a second order phase transition and in the supercritical regime, the zero fixed point is unstable and there is a unique stable nontrivial fixed point. By contrast, systems with cooperative branching and deaths have a first order phase transition and in the supercritical regime, there are two stable fixed points (one zero, the other nonzero) separated by an unstable fixed point.

For the contact process, it is known that the behavior of spatial models is similar to the mean-field model. In particular, in the supercritical regime, there is a unique nontrivial homogeneous invariant law, which is the limit law started from any nontrivial homogeneous initial law.

For systems with cooperative branching, there are so far only a few rigorous results. It is believed that the behavior of spatial models is subtle in the sense that sometimes, such models behave as suggested by the mean-field model, but sometimes, by contrast, they behave more similar to the contact process. In particular, it seems likely this is the case for large branching rates, as formulated in the following problem:

Problem 2 *For systems with cooperative branching with sufficiently large branching rates, prove that there is a unique nontrivial invariant law, which is the long-time limit law of the process conditioned on survival.*

By contrast, on sufficiently high dimensional lattices such as trees, it seems the model behaves more like its mean-field model, which suggests the following problem:

Problem 3 *For systems with cooperative branching on regular trees, show that there is an intermediate parameter range where finite systems die out a.s. while there are two nontrivial translation-invariant stationary laws.*

A key tool in proving that the contact process has a unique nontrivial homogeneous invariant law is additive systems duality [Gri79], which relates these processes to oriented percolation. The duality theory for systems with cooperative branching and deaths has been studied by Gray [Gra86] and more in detail also by Sturm and Swart [SS18]. Here the open paths of additive particle systems need to be replaced by more complex structures, that can also branch. The mean-field limit of the dual process was studied in [BCH18, MSS18b]. Problems 2 and 3 are nice challenges at the edge of the modern state-of-the-art concerning duality techniques in interacting particle systems.

III. DYNAMICAL RECURSIVE TREE PROCESSES

A *Recursive Tree Process* (RTP) is a random rooted tree in which each vertex has a state that is a random function of its descendants. In particular, for a finite RTP, the states of all internal vertices can inductively be calculated from the states of the leaves and the random functions attached to the internal vertices. RTPs arise naturally in the study of mean-field interacting particle systems and their duals [MSS18b]. In this case, the random tree is the family tree of a branching process and the random functions attached to the internal vertices are i.i.d. Fixed points of the mean-field equation give rise to stationary RTPs that can be infinitely large, if the corresponding branching process survives.

An interesting and subtle question is whether such a stationary RTP is *endogenous*. This term, introduced in [AB05], means that the state of the root is a.s. uniquely determined by the random functions attached to the vertices of the tree. There exist currently two approaches to determine whether a given stationary RTP is endogenous.

The first approach, due to [AB05] (and slightly improved in [MSS18a]) links endogeny to *bivariate uniqueness*. In short, [AB05] proved that if an RTP is nonendogenous, then it should be possible couple two RTPs so that they are described by the same random tree and random functions attached to the vertices, but the state at the root is with positive probability different. This technique has been used in to prove nonendogeny of frozen percolation on the binary tree [RST19] and of certain systems with cooperative branching [MSS18b].

The second approach, pioneered in [War06] links nonendogeny to the existence of a *dynamical RTP* where the random tree and random functions are again fixed but the states at all vertices evolve in time in a nontrivial way. This approach links endogeny to a certain Perron-Frobenius eigenvalue being > 1 or < 1 , with partial results in the critical case $= 1$. In [War06], several simplifying assumptions are made: the tree is binary, the functions assigned to vertices are symmetric, and each vertex can assume only finitely many states. These assumptions so far severely limit the applicability of the methods. A first step would be to solve the following problem:

Problem 4 *Extend the results of [War06] to trees that are not binary and to asymmetric functions, and resolve what happens in the critical case.*

A further subject to be investigated is the link between the approach of [War06] and results of [JPS19], who link nonendogeny to survival of a branching process of pivotal vertices. It seems the Perron-Frobenius eigenvalue that governs the survival of this branching process is the same as in [War06], but this needs to be established more precisely.

IV. PERIODIC BEHAVIOR OF INTERACTING PARTICLE SYSTEMS

A large part of the study of interacting particle systems is devoted to studying invariant measures and their domains of attraction. In principle, however, it is possible for an interacting particle system not to converge to an invariant measure at all. One way in which this can happen is through periodic behavior.

There are several known examples of interacting particle systems that in the mean-field limit exhibit periodic behavior [CFT16, CDFT20]. On the other hand, that truly spatial interacting particle systems can exhibit periodic behavior has been proved only recently by Jahnke and Külske [JK14]. Although a major milestone, this paper leaves many questions open. For example, although it proves the existence of an interacting particle systems with periodic behavior, it does not have finite range interactions and the authors do not even explicitly describe its dynamics. Also, it is not known what initial conditions lead to periodic behavior. An important problem is thus:

Problem 5 *Give an explicit construction of an interacting particle system with finite range interactions exhibiting periodic behavior.*

The construction of [JK14] works only in dimensions three and more. An older result of Mountford [Mou95] shows that one-dimensional interacting particle systems cannot exhibit periodic behavior. Whether periodic behavior is possible in two dimensions is still open. Thus, a second problem is:

Problem 6 *Determine whether two dimensional interacting particle systems can exhibit periodic behavior.*

V. RANK-BASED PARTICLE SYSTEMS

A number of authors, motivated by various problems in real life, have introduced stochastic particle models where particles interact only through their relative order, usually in the sense that in certain situations, the highest (or lowest) particle is removed. This includes the famous Bak-Sneppen model [BS93] for evolution (survival of the fittest), the Stigler-Luckock model [Sti64, Luc03] in financial mathematics (execution of the best buy/ask offer), certain queueing models [CG09] (execution of the most urgent task), and a model loosely inspired by canyon formation [Swa17].

In the Stigler-Luckock, buy and sell limit orders are placed in an order book according to independent Poisson processes. Buy (resp. sell) limit orders that arrive above (resp. below) the current ask (resp. bid) price are immediately matched to the best available opposite order, which results in both orders being removed from the order book. In simulations, one observes the formation of a “competitive window” which is an interval such that all orders arriving in this interval are eventually matched, but in the long run, orders arriving outside this interval are never matched and stay in the order book forever. Luckock [Luc03] used nonrigorous methods to derive a formula for the boundaries of the competitive window and the equilibrium distributions of the bid and ask price. These formulas were partially made rigorous in [KY18, Swa18a], but the following problem remains open:

Problem 7 *Starting from an empty order book, show that the distributions of the bid and ask price converge to the equilibrium distributions predicted by Luckock.*

In [FS16], Formentin and Swart studied a queueing model similar to the one in [CG09]. In this model, emails arrive into someone’s inbox with Poisson rate λ_{in} and are answered with Poisson rate $\lambda_{\text{out}} < \lambda_{\text{in}}$. Upon arrival, each incoming email is assigned a priority that is uniformly distributed on $[-\lambda_{\text{in}}, 0]$ and the owner of the email always answers the email with the highest priority. The result of this is that emails with a priority below $-\lambda_{\text{out}}$ stay in the inbox forever. The paper [FS16] proves an asymptotic formula for distribution of emails waiting to be answered with priorities just above $-\lambda_{\text{out}}$, linking it to the convex hull of Brownian motion. The following problem is open:

Problem 8 *Show that the asymptotic formula of [FS16] also holds for the Stigler-Luckock model and for the model for canyon formation in [Swa17].*

VI. SHARPNESS OF THE PHASE TRANSITION

It is known that the connection probability in percolation decays exponentially fast in the whole subcritical regime. This was first proved by Menshikov [Men86] and then by Aizenman and Barsky [AB87], who used Russo’s formula and a system of two differential inequalities. Bezuidenhout and Grimmett [BG91] adapted their methods to oriented percolation and the contact process. Duminil-Copin and Tassion [DT16] gave a much simplified proof for both unoriented and oriented percolation that involves only a single differential inequality. Swart [Swa18b] gave a new, simple proof for the contact process that surprisingly does not use Russo’s formula or differential inequalities but instead is based on a special harmonic function discovered in [Swa09]. Several questions remain open:

Problem 9 *Can the method of [DT16] be adapted to the contact process?*

Problem 10 *Can the methods of [DT16] and [Swa18b] be adapted to more general monotone interacting particle systems?*

For the contact process, there are in general two critical values for the infection rate, λ_{glob} and λ_{loc} , which are related to global and local survival. It is known that $\lambda_{\text{glob}} = \lambda_{\text{loc}}$ for the contact process on \mathbb{Z}^d but $\lambda_{\text{glob}} < \lambda_{\text{loc}}$ for the contact process on regular trees [Lig99]. The following problem is open:

Problem 11 *For the contact process on general lattices, show that local survival is exponentially fast in the whole regime $\lambda < \lambda_{\text{loc}}$.*

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