Computerized adaptive testing using Bayesian networks

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- ETS has a research unit doing research on adaptive testing using Bayesian networks: `http://www.ets.org/research/`

Rasch model (G. Rasch,1960)

Model variables

$Y_{n,i}$	binary response variable - its values indicates whether the answer of person n to question i was correct
$n = 1, \ldots, N$	person index
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Model parameters

δ_i	difficulty of question i - fixed effects
β_n	ability (knowledge level) of person n - a random effect

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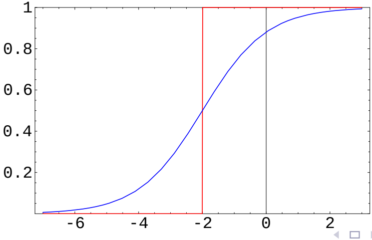
$P(Y_{n,i} = 1 \mid \beta_n)$ for $\delta_i = -2$



Probability distribution for random effect β_n

$$P(\beta_n) \quad = \quad \mathcal{N}(0, \sigma^2)$$

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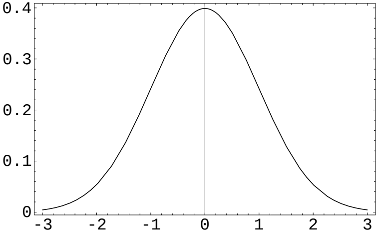
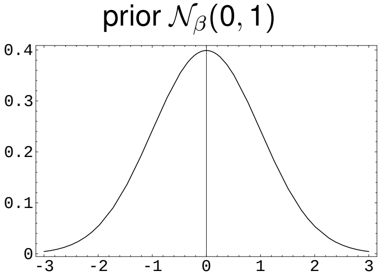


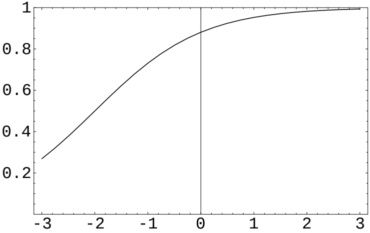
Computations with the Rasch model



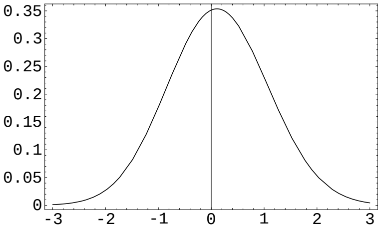
$\mathcal{N}_\beta(0,1)$

Computations with the Rasch model

$P(Y = 1 \mid \beta, \delta_1 = -2)$



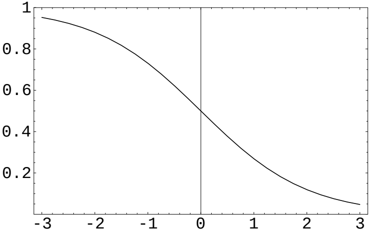
posterior



$$\mathcal{N}_\beta(0, 1) \cdot P(Y = 1 \mid \beta, \delta_1 = -2)$$

Computations with the Rasch model

$P(Y = 0 \mid \beta, \delta_2 = 0)$



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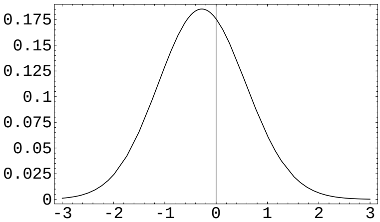

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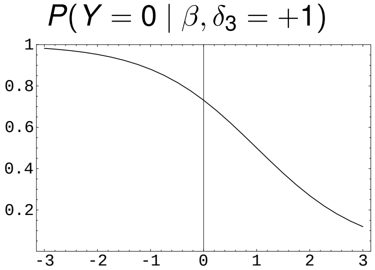

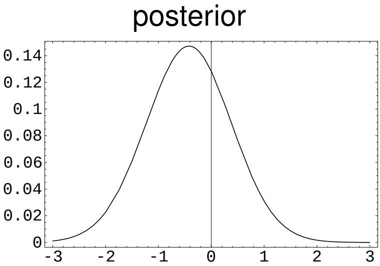

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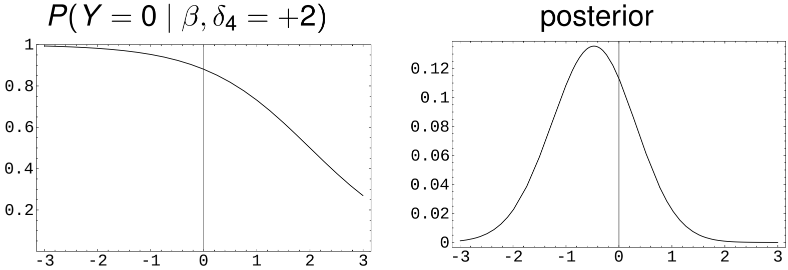

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Student model and evidence models (R. Almond and R. Mislevy, 1999)

- The variables of the models are: (1) skills, abilities, misconceptions, etc. - for brevity called skills - the vector of skills is denoted $\boldsymbol{S}$ and (2) items (questions) - the vector of questions is denoted $\boldsymbol{X}$.

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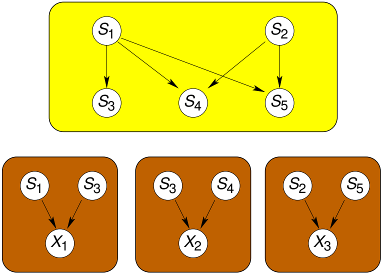
- **student model** describes relations between student skills.
- **evidence models** - one for each item (question) - describes relations of the item to the skills.



Building Bayesian network models

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- Discussions with **domain experts**: expert knowledge is used to get the structure and parameters of the model
- A dataset of records is collected and a **machine learning** method is used to to construct a model and estimate its parameters.
- A **combination** of previous two: e.g. experts suggest the structure and collected data are used to estimate parameters.

Example of a simple diagnostic task

We want to diagnose presence/absence of three skills

$$S_1, S_2, S_3$$

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The questions depend on skills and their dependence is described by conditional probability distributions

$$P(X_{i,j} = 1 | S_i = s_i, S_j = s_j) \quad = \quad \begin{cases} 1 & \text{if } (s_i, s_j) = (1, 1) \\ 0 & \text{otherwise.} \end{cases}$$

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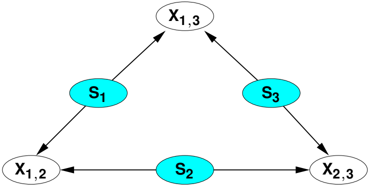
Assume all answers were wrong, i.e.,

$$X_{1,2} = 0, \quad X_{1,3} = 0, \quad X_{2,3} = 0 \quad .$$

Reasoning under the assumption of skills' independence



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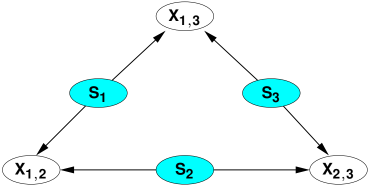

First, assume the skills are pairwise independent, i.e.,

$$P(S_1, S_2, S_3) = P(S_1) \cdot P(S_2) \cdot P(S_3)$$

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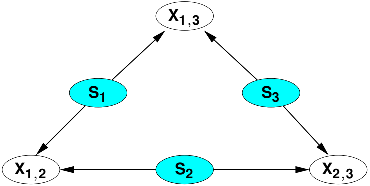

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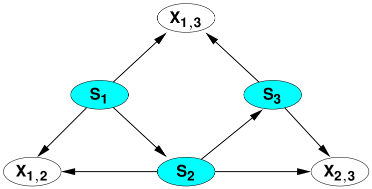
Then conditional probabilities for $j = 1, 2, 3$ are

$$P(S_j = 0 \mid X_{1,2} = 0, X_{1,3} = 0, X_{2,3} = 0) = 0.75 \quad ,$$

i.e., we cannot decide with certainty, which skills are present and which are absent.

Modeling dependence between skills



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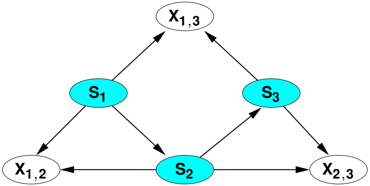

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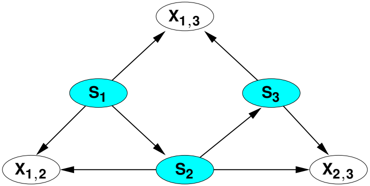

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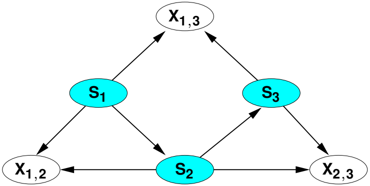

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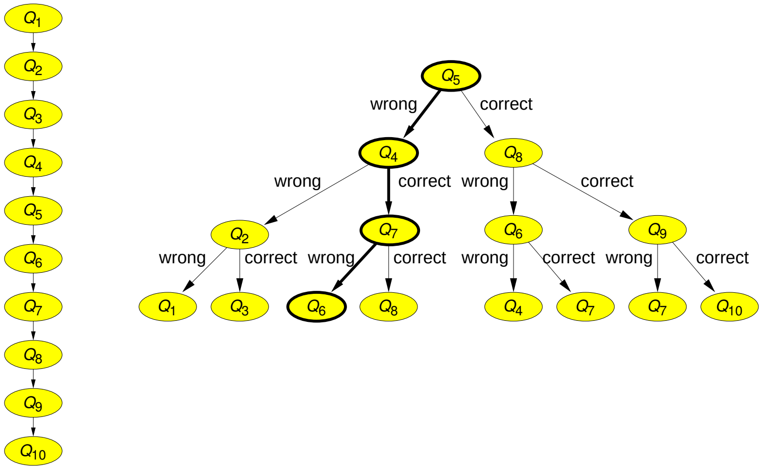
For $i = 1, 2, 3$: $P(S_i \mid X_{1,2} = 0, X_{1,3} = 0, X_{2,3} = 0) = P(S_i \mid X_{2,3} = 0)$.
$X_{2,3} = 0$ gives **the same information** as $X_{1,2} = X_{1,3} = X_{2,3} = 0$.


Q_1
Q_2
Q_3
Q_4
Q_5
Q_6
Q_7
Q_8
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Q_{10}
Q_5
wrong
correct
Q_4
Q_8
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correct
wrong
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Q_2
Q_7
Q_6
Q_9
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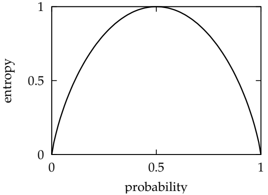

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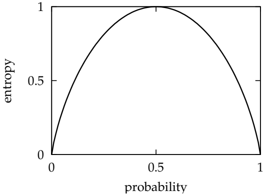
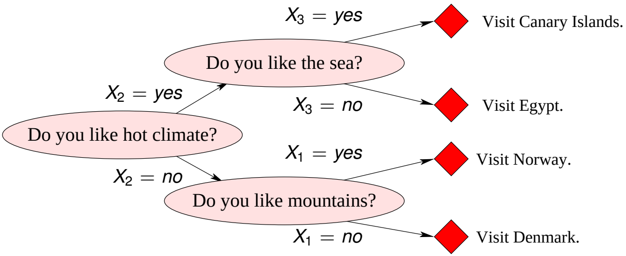


- "The lower the entropy the more we know."

Building strategies using the models



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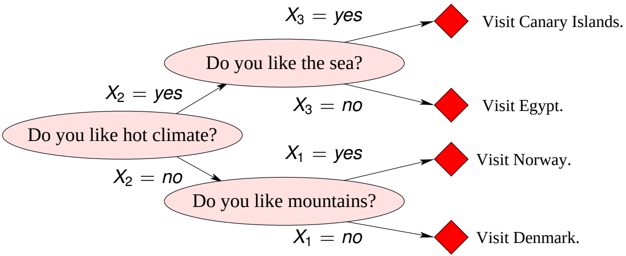

For all terminal nodes (leaves) $\ell \in \mathcal{L}(\boldsymbol{s})$ of a strategy $\boldsymbol{s}$ we define:

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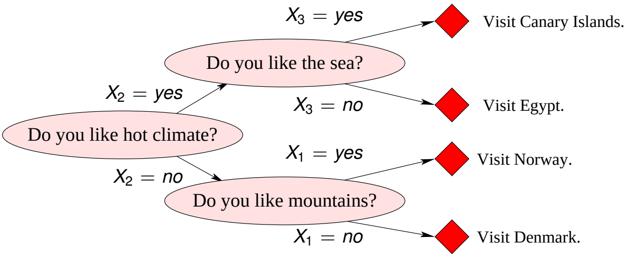

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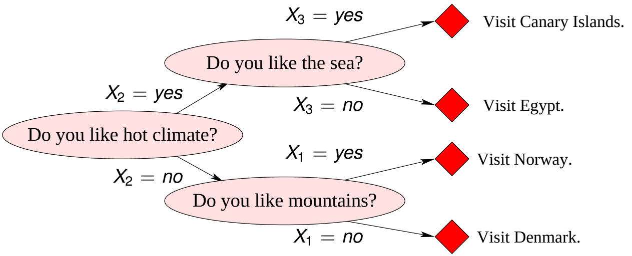

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- Also during the process, when we collected evidence $\boldsymbol{e}$, we update the probability of getting to a terminal node to conditional probability $P(\boldsymbol{e}_\ell | \boldsymbol{e})$

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- Specifically, we will maximize the expected information gain,

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- The evaluation function can be, e.g., the information gain.
- Information gain in a node n of a strategy
  $IG(\boldsymbol{e}_n) = H(P(\boldsymbol{S})) - H(P(\boldsymbol{S} \mid \boldsymbol{e}_n))$

For each strategy we can compute:

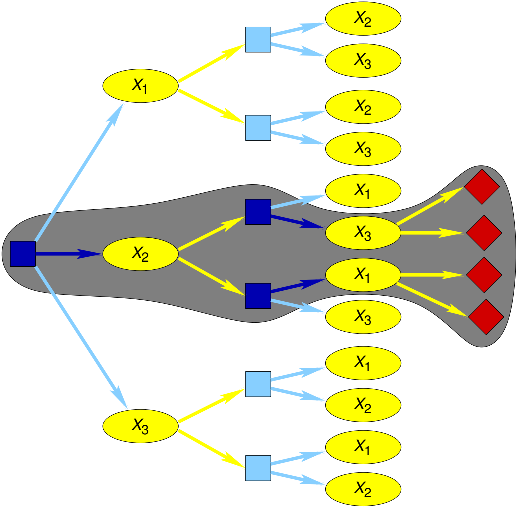
- expected value of the strategy:

$$E_f(\boldsymbol{s}) = \sum_{\ell \in \mathcal{L}(\boldsymbol{s})} P(\boldsymbol{e}_\ell) \cdot f(\boldsymbol{e}_\ell)$$

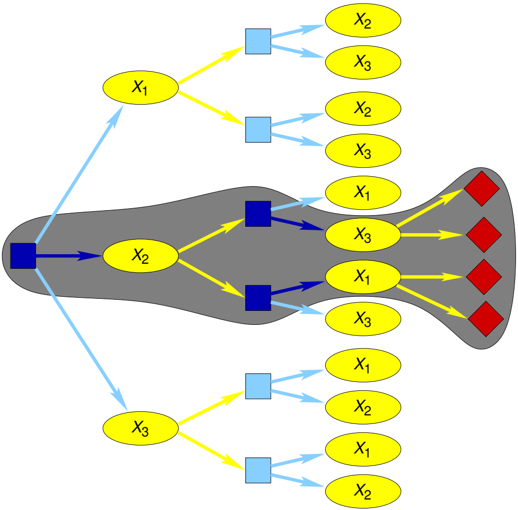
The goal is

- to find a strategy that maximizes its expected value.
- Specifically, we will maximize the expected information gain,
- which corresponds to minimization of expected entropy.

Space of tests of length 2 of 3 possible questions



Space of tests of length 2 of 3 possible questions



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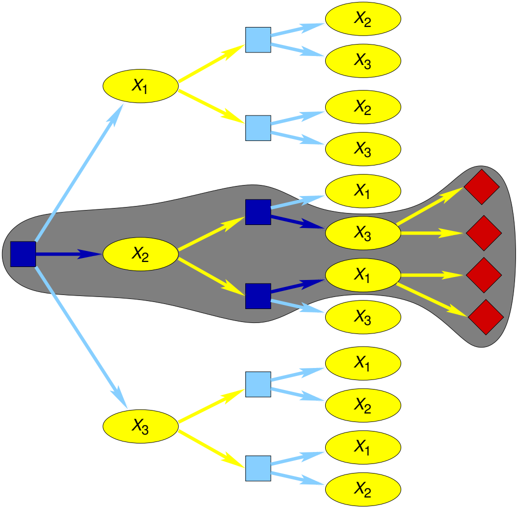
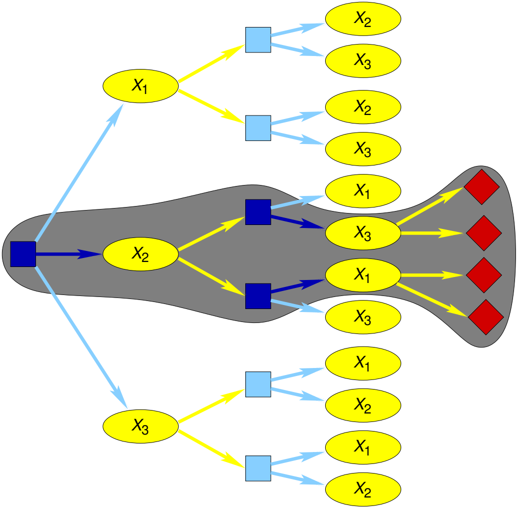
  $H(\boldsymbol{e}_n) = H(P(\boldsymbol{S} \mid \boldsymbol{e}_n))$

Space of tests of length 2 of 3 possible questions



- **Entropy** in node n:
  $H(\boldsymbol{e}_n) = H(P(\boldsymbol{S} \mid \boldsymbol{e}_n))$
- **Expected entropy** of a test $\boldsymbol{t}$
  $E_H(\boldsymbol{t}) = \sum_{\ell \in \mathcal{L}(\boldsymbol{t})} P(\boldsymbol{e}_\ell) \cdot H(\boldsymbol{e}_\ell)$

Space of tests of length 2 of 3 possible questions



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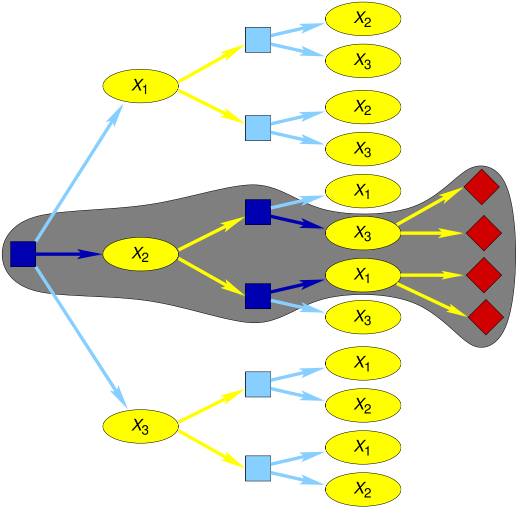
Space of tests of length 2 of 3 possible questions



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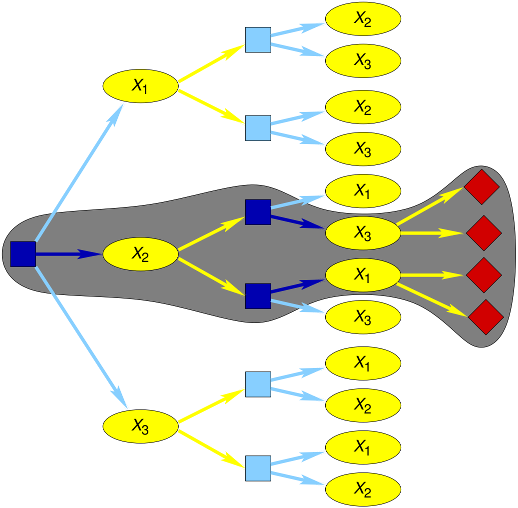

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  $\boldsymbol{t}^\star = \arg\min_{\boldsymbol{t} \in \mathcal{T}} E_H(\boldsymbol{t})$.
- **Myopically optimal test**:
  in each step a we select a question that minimizes expected entropy of the test that would terminate after this question (one step look ahead).

Adaptive test of basic operations with fractions

Examples of tasks:

$$
\begin{array}{llcl}
T_1: & \left(\frac{3}{4} \cdot \frac{5}{6}\right) - \frac{1}{8} & = & \frac{15}{24} - \frac{1}{8} = \frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \\
T_2: & \frac{1}{6} + \frac{1}{12} & = & \frac{2}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4} \\
T_3: & \frac{1}{4} \cdot 1\frac{1}{2} & = & \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8} \\
T_4: & \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{3} + \frac{1}{3}\right) & = & \frac{1}{4} \cdot \frac{2}{3} = \frac{2}{12} = \frac{1}{6} \quad .
\end{array}
$$

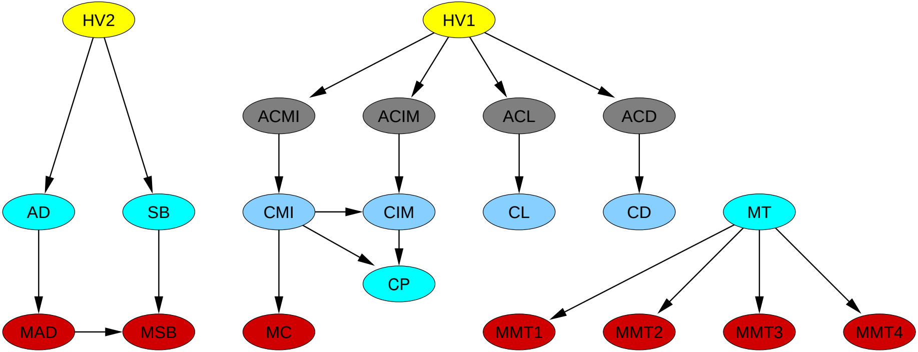
Elementary and operational skills

CP	Comparison (common numerator or denominator)	$\frac{1}{2} > \frac{1}{3}, \quad \frac{2}{3} > \frac{1}{3}$
AD	Addition (comm. denom.)	$\frac{1}{7} + \frac{2}{7} = \frac{1+2}{7} = \frac{3}{7}$
SB	Subtract. (comm. denom.)	$\frac{2}{5} - \frac{1}{5} = \frac{2-1}{5} = \frac{1}{5}$
MT	Multiplication	$\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$
CD	Common denominator	$\left(\frac{1}{2}, \frac{2}{3}\right) = \left(\frac{3}{6}, \frac{4}{6}\right)$
CL	Canceling out	$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{3}$
CIM	Conv. to mixed numbers	$\frac{7}{2} = \frac{3 \cdot 2 + 1}{2} = 3\frac{1}{2}$
CMI	Conv. to improp. fractions	$3\frac{1}{2} = \frac{3 \cdot 2 + 1}{2} = \frac{7}{2}$

Misconceptions

Label	Description	Occurrence
MAD	$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$	14.8%
MSB	$\frac{a}{b} - \frac{c}{d} = \frac{a-c}{b-d}$	9.4%
MMT1	$\frac{a}{b} \cdot \frac{c}{b} = \frac{a \cdot c}{b}$	14.1%
MMT2	$\frac{a}{b} \cdot \frac{c}{b} = \frac{a+c}{b \cdot b}$	8.1%
MMT3	$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$	15.4%
MMT4	$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b+d}$	8.1%
MC	$a\frac{b}{c} = \frac{a \cdot b}{c}$	4.0%

Student model



Evidence model for task $T1$

$$\left(\frac{3}{4} \cdot \frac{5}{6}\right) - \frac{1}{8} = \frac{15}{24} - \frac{1}{8} = \frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

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$$T1 \quad \Leftrightarrow \quad MT \,\&\, CL \,\&\, ACL \,\&\, SB \,\&\, \neg MMT3 \,\&\, \neg MMT4 \,\&\, \neg MSB$$

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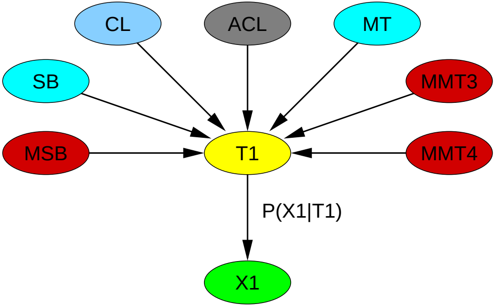
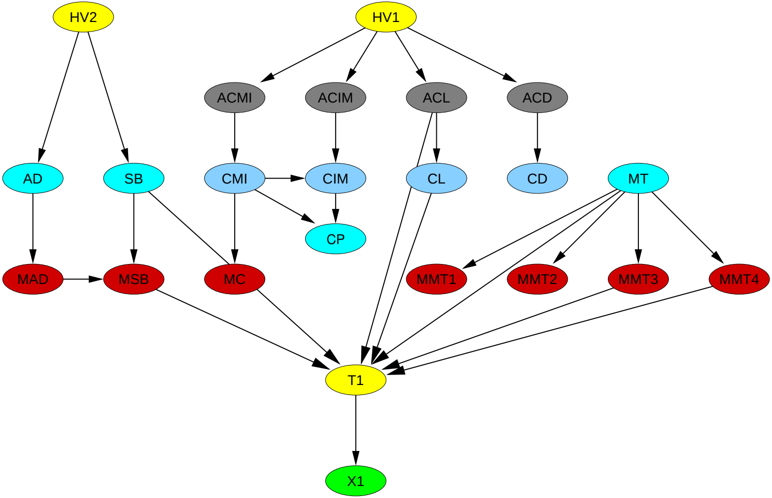
$$\left(\frac{3}{4}\cdot\frac{5}{6}\right)-\frac{1}{8}=\frac{15}{24}-\frac{1}{8}=\frac{5}{8}-\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

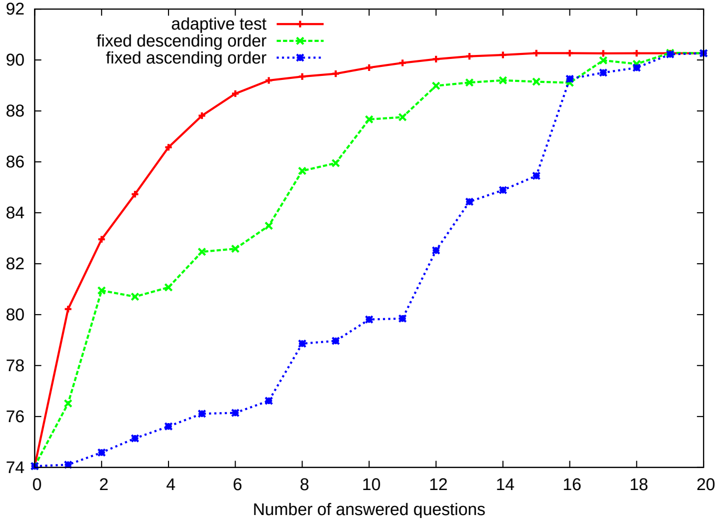
$$T1 \quad \Leftrightarrow \quad MT \ \& \ CL \ \& \ ACL \ \& \ SB \ \& \ \neg MMT3 \ \& \ \neg MMT4 \ \& \ \neg MSB$$



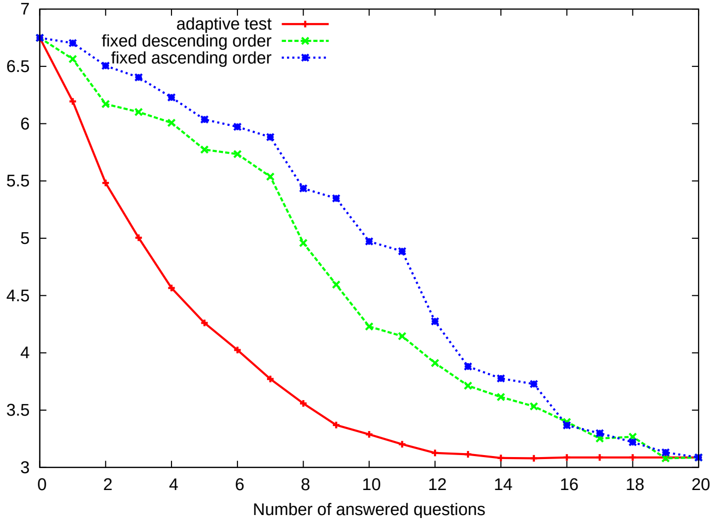
Evidence model for task $T1$ connected with the student model



Skill Prediction Quality



Entropy



7
6.5
6
5.5
5
4.5
4
3.5
3
0
2
4
6
8
10
12
14
16
18
20
adaptive test
fixed descending order
fixed ascending order
Number of answered questions

Conclusions and References

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- J. Vomlel: Bayesian networks in educational testing, International Journal of Uncertainty, Fuzziness and Knowledge Based Systems, Vol. 12, Supplementary Issue 1, 2004, pp. 83—100.